



**Philosophia Scientiæ**

Travaux d'histoire et de philosophie des sciences

**17-1 | 2013**

**The Epistemological Thought of Otto Hölder**

---

## Geometry and Measurement in Otto Hölder's Epistemology

**Paola Cantù**

---



**Electronic version**

URL: <http://journals.openedition.org/philosophiascientiae/832>

DOI: 10.4000/philosophiascientiae.832

ISSN: 1775-4283

**Publisher**

Éditions Kimé

**Printed version**

Date of publication: 1 March 2013

Number of pages: 131-164

ISBN: 978-2-84174-620-0

ISSN: 1281-2463

**Electronic reference**

Paola Cantù, « Geometry and Measurement in Otto Hölder's Epistemology », *Philosophia Scientiæ* [Online], 17-1 | 2013, Online since 01 March 2016, connection on 02 November 2020. URL : <http://journals.openedition.org/philosophiascientiae/832> ; DOI : <https://doi.org/10.4000/philosophiascientiae.832>

---

Tous droits réservés

# Geometry and Measurement in Otto Hölder's Epistemology\*

Paola Cantù

Aix-Marseille Université – CNRS, CEPERC, UMR 7304,  
Aix-en-Provence (France)

**Résumé :** L'article a pour but d'analyser la conception de la géométrie et de la mesure présentée dans *Intuition et Raisonnement* [Hölder 1900], « Les axiomes de la grandeur et la théorie de la mensuration » [Hölder 1901] et *La Méthode mathématique* [Hölder 1924]. L'article examine les relations entre a) la démarcation introduite par Hölder entre géométrie et arithmétique à partir de la notion de 'concept donné', b) sa position philosophique par rapport à l'apriorisme kantien et à l'empirisme et c) le choix du postulat de la continuité de Dedekind parmi les axiomes de la théorie des grandeurs. L'article montre que les choix faits au niveau axiomatique sont reliés au cadre épistémologique et que l'originalité de Hölder consiste surtout dans son analyse au cas par cas des procédures déductives en mathématiques.

**Abstract:** The aim of the paper is to analyze Hölder's understanding of geometry and measurement presented in *Intuition and Reasoning* [Hölder 1900], "The Axioms of Quantity and the Theory of Measurement" [Hölder 1901], and *The Mathematical Method* [Hölder 1924]. The paper explores the relations between a) Hölder's demarcation of geometry from arithmetic based on the notion of given concepts, b) his philosophical stance towards Kantian apriorism and empiricism, and c) the choice of Dedekind's continuity in the axiomatic formulation of the theory of magnitudes. The paper shows that the choices made at the axiomatic level reflect Hölder's epistemological framework, and individuates the originality of his approach in the case by case analysis of deductive procedures.

---

\* The author would like to thank two anonymous referees for their valuable constructive criticism of this paper, and Volker Peckhaus, Oliver Schlaudt, Philippe Nabonnand, Gerhard Heinzmann together with all the participants of the Workshop on Hölder held in Nancy in 2010 for their precious comments in relation to a preliminary version.

# 1 Introduction

Otto Hölder is known for his results on the theory of groups (see [Gray 1994], [Schlimm 2008]) and for the paper on the axiomatic theory of measurement published in 1901 under the title “The Axioms of Quantity and the Theory of Measurement” (see [Diez 1997], [Darrigol 2003], [Michell 1993]). Less investigated are the epistemological considerations on the relations between geometry, arithmetic and theory of measurement that Hölder developed before and after this paper: most of them are contained either in the paper *Intuition and Reasoning* [Hölder 1900] or in the volume *The Mathematical Method* [Hölder 1924].<sup>1</sup> Further remarks on indirect proofs (see [Hölder 1929], [Hölder 1930]), fallacies of circularity in mathematics (see [Hölder 1926]), the axiomatization of mathematics (see [Hölder 1931], [Radu 2003]), and the paradoxes are contained in several minor articles published between 1926 and 1931.

## 1.1 Continuity, deduction and axiomatics

Hölder’s epistemology is centered on the investigation of the notion of deduction: its relation to geometry was first investigated in *Intuition and Reasoning* [Hölder 1900] and further developed in *The Mathematical Method* [Hölder 1924], which is explicitly presented as a critical reaction to Russell, Couturat and Natorp. The paper investigates some historical and philosophical questions about the distinction between geometry and arithmetic, deduction and experience, synthetic and given. 1) Does Hölder endorse the Grassmannian distinction between formal and empirical sciences and agree with Gauß and Helmholtz about the empirical nature of geometry? 2) How does Hölder combine a rather empirical conception of geometry with the Kantian terminology (‘synthetic’ opposed to ‘given’) used to describe it? 3) Are the primitive concepts of the theory of measurable magnitudes given or constructed? What is the status of the Archimedean axiom?

Hölder’s approach was devoted to the investigation of the possible relations between deduction and experience—a classical epistemological problem that he investigated in an original perspective, i.e., on the basis of a case by case analysis of the use of deduction in several mathematical disciplines: geometry, mechanics, measurement theory, the theory of continuum, analytical geometry, the theory of manifolds, the theory of integration and differentiation, the theory of real and imaginary numbers, and the theory of prime numbers. In modern terminology, one could say that Hölder was particularly interested in *mathematical practice*, believing that the logical investigation of the mathematical method should not precede but follow the analysis of math-

---

1. Given that there is an English translation of the 1899 paper published in this issue but no English translation of the 1924 volume, I will often quote more extensively from the latter rather than from the former. Translations from [Hölder 1924] are mine.

ematical theories and examples, and should not limit itself to number theory, but include all relevant mathematical disciplines [Hölder 1924, 1–3].

This case by case investigation of mathematics, developed during the First World War and published with minor changes more than ten years later, is based on the idea that logic might be field-dependent, i.e., that a general logic might only arise from the investigation of the deductive procedures used in specific mathematical disciplines.<sup>2</sup> Hölder, unlike most of his contemporaries, claimed that if logic is the science of deduction, then it cannot itself be built deductively [Hölder 1924, 2–3, 277], and that symbolic language, although useful to express mathematics, is not as useful in logic, where new symbols and new concepts are often introduced, requiring at each time new linguistic clarifications and comments [Hölder 1924, 5]. The ordinary language, thanks to its tonality and suggestive power, was thus praised as the language best suited to investigate deduction [Hölder 1924, 7–8].

According to Hölder, an essential characteristic of mathematics is that it allows the construction of concepts: the aim of his logic of mathematics is thus the investigation of different ways in which concepts might be constructed (as pure synthetic concepts in arithmetic and as hypothetico-synthetic concepts in geometry). The notion of deduction is related to the mathematical activity of construction of concepts, and thus to the Kantian perspective, which is the object of an internal criticism. The main question is still Kantian: “how can pure mathematics be possible?”, but the border between *a priori* and non *a priori* knowledge is differently drawn: geometry does not belong to pure mathematics [Hölder 1924, 5–8].

If geometry is a science whose primitive axioms have their origin in experience, and measurement theory, like geometry, “can be based upon a set of facts”, why does the latter include among its axioms the postulate of Dedekind's continuity? The latter is a consequence of the commonly accepted axiom of Archimedes, but in the 1901 paper, Hölder had developed an original example of a non-Archimedean continuum (containing infinitely great and small quantities): why did he not further develop this intuition? To understand Hölder's axiomatic choices, a deep understanding of the peculiarity of his epistemological framework is no less useful than a clear understanding of the technical differences between his model and other contemporary approaches to non-Archimedean continuity by Bettazzi and Veronese.

## 1.2 Plan of the paper

In section 2, I will briefly sketch Hölder's conception of ‘given’ concepts in geometry. Hölder's epistemology was deeply influenced by the debate on the relationship between geometry and experience raised by Gauß, Graßmann and

---

2. Among the precursors of this enterprise, Hölder quotes Lambert's *New Organon* [Lambert 1764], Pasch's *Lectures on Modern Geometry* [Pasch 1882] and Hilbert's *Foundations of Geometry* [Hilbert 1899].

Helmholtz, but also by some remarks by Konrad Zindler, a disciple of Meinong. Hölder's epistemology can be better understood in the light of his interest for deduction and for case by case analysis of mathematical practice, which led him to oppose logicism as well as Kantianism, but also to discuss some limits of empiricism.

In section 3, I will discuss whether Hölder might be considered as a follower of Kant's philosophy of mathematics and at the same time a supporter of Pasch's empiricism. In particular, I will focus my attention on the reasons why Hölder used a Kantian terminology ('synthetic' as opposed to 'given') even if he criticized Kant's conception. I will claim that in around 1900 Hölder was surely nearer to Pasch's empirical understanding of mathematics and to Helmholtz's epistemology rather than to Kant's philosophy, even if he did not consider the Kantian position as provably wrong, and adopted a terminology that became more Kantian in his later work (see for example the distinction 'given'–'constructed' being substituted by the distinction 'pure synthetic'–'hypothetico-synthetic').

In section 4, I will discuss the relations between this idea of 'given' and Hölder's conception of continuity, which is at the base of his theory of measurement. Hölder's theory of measurable magnitudes is quite different from the theories developed by Veronese, Bettazzi and Hilbert: it is Archimedean, and even Dedekind-continuous. I will discuss the role of the Archimedean axiom and of Dedekind's postulate with respect to experience.

In section 5, I will outline some complex relations between Hölder's epistemology, as it emerges from the reconstruction made in section 2 and section 3, and the conceptions of other 19th century mathematicians and philosophers. The originality of Hölder's epistemological approach to geometry and measurement will thus appear more clearly.

## 2 A demarcation between arithmetic and geometry

Hölder's philosophy of mathematics is based on the "traditional" idea that there is a fundamental difference between arithmetic and geometry. This idea is traditional inasmuch as it was introduced by the Greeks and maintained unaltered up to late modernity, when the development of algebra began to outline correspondences and similarities between discrete and continuous quantities. The 19th century, from Gauß to Helmholtz, was the time for the introduction of a new distinction, no longer based on a qualitative distinction between the two disciplines—the former being the science of (discrete) plurality and the second the science of (continuous) magnitude—but rather on their relationship to experience. The former is a pure science based on definitions, while the latter is a natural science (it is part of physics) and based on axioms. Once the distinction is related not to the kind of objects the sciences are applied

to, but to a difference in the origin of their concepts and in their formulation, the investigation of the relations between intuition, experience and mathematical rigor in geometrical deductions becomes the most relevant epistemological question. Hölder introduced a shift, separating the question of the origin of the concepts, which he considered as a question that pertains to philosophy of mathematics, from the question of the demarcation between arithmetic and geometry. The latter is essentially a mathematical question and can be answered by a thorough consideration of several examples of primitive concepts with respect to the activity of construction needed to introduce them.

## 2.1 What is 'given'?

The idea that geometrical objects and properties are 'given' and not constructed by our thought goes back to Gauß and Graßmann. It is well known that Gauß remarked that space also has a reality outside our mind, and that we cannot describe *a priori* the laws of that reality, at least not entirely *a priori*.<sup>3</sup> Graßmann distinguished formal and real sciences, according to the fact that they investigate a being that is posited from thought itself or that faces thought as something autonomous.<sup>4</sup> Given objects are objects that confront our thought and whose properties are independent from our mental activities. This conception of 'given' as some physical reality that exists independently and autonomously from our thought is often associated in the 19th century to the idea that geometry is an empirical science, and more precisely a part of physics (Gauß, Graßmann, Helmholtz) or at most a mixed science, i.e., partly empirical and partly formal (Veronese).

In *Intuition and Reasoning* Hölder held a view that does not entirely agree with Gauß's perspective: 1) firstly, he used the term 'given' in a different sense: this results in the fact that not all geometrical concepts are given; 2) secondly, even when he discussed the philosophical question of the origin of 'given', he remarked that it might originate either in experience or in intuition, and did not explicitly refute the Kantian conception.

### 2.1.1 Three meanings of the term 'given'

The term 'given' occurs very often in *Intuition and Reasoning*, but its meaning is not always clear. Sometimes it refers to concepts that "are available to the

---

3. See the letters to Bessel dated 9 April 1830 and 27 January 1829. In the latter Gauß remarked that his belief that geometry cannot be grounded entirely *a priori* had become stronger [Gauß 1900, vol. VIII, 200–201].

4. "The principal division of the sciences is into the real and the formal. The real represent the existent in thought as existing independently of thought, and their truth consists in the correspondence of the thought with that existent. The formal on the other hand have as their object what has been produced by thought alone, and their truth consists in the correspondence between the thought processes themselves" [Graßmann 1844, xxi, Eng. trans., 23].

geometer as a given material”,<sup>5</sup> sometimes it refers to concepts that are not “defined by means of a construction”,<sup>6</sup> sometimes to concepts that are “given within intuition”,<sup>7</sup> sometimes to concepts that are not constructed but “taken from intuition or experience”.<sup>8</sup>

To understand Hölder’s remarks, I will first distinguish between three possible meanings of the term ‘given’: 1) epistemological, 2) axiomatic, 3) genetic.

1. According to the epistemological meaning, something ‘given’ is something that is epistemologically accessible via experience or intuition, i.e., by means of our senses—be they exterior or inner. In *Intuition and reasoning*, Hölder remarked that what is given in geometry might be either given empirically or given within intuition, but he rather preferred the first option, which guarantees more freedom—claiming that philosophers who consider geometrical concepts as given within intuition tend on the contrary to criticize the choice of non intuitive concepts as primitive concepts [Hölder 1900, 11, Eng. trans. 20]. There is no one-to-one correspondence between ‘given’ and primitive concepts, because some primitive concepts might be constructed rather than ‘given’ in intuition or in experience. Moreover, some “concepts which were clearly immediately abstracted from intuition, can later be deduced” [Hölder 1900, 15, Eng. trans. 23].<sup>9</sup> This notion of ‘given’ is an epistemological distinction that is relative to the state of our knowledge; it is thus different from the logical notion of primitiveness used in modern axiomatics.<sup>10</sup>
2. According to the axiomatic meaning, ‘given’ refers to concepts that occur in the primitive propositions of a mathematical theory, whatever the epistemological access we might have to them, or whatever the kind of

---

5. “Beside these concepts, which are available to the geometer as a *given* material, some assertions of a more general kind (which hold or should hold for these concepts) are also established from the beginning; these fundamental facts, being presupposed without proof, are called axioms or postulates” [Hölder 1900, 1, Eng. trans., 16]. Italics in this and in the following two quotations are mine.

6. “Some concepts are defined by means of a construction, others are basically taken as *given*” [Hölder 1900, 1, Eng. trans., 15].

7. “[Philosophers] consider Euclid’s axiom of parallel lines as necessarily *given* within intuition, and consider any geometry that operates intuitively with some other assumption as an absurdity” [Hölder 1900, 6, Eng. trans., 20].

8. “We distinguished in geometry between primary concepts (taken from intuition or experience) on the one hand, and geometrically constructed concepts on the other hand” [Hölder 1900, 15, Eng. trans., 23].

9. Hölder refers here to the similitude of figures, which can be either intuited and used to ground the concept of ratio, or derived from the theory of proportions, as Euclid did. Even if “it may be taken as an immediate intuitive fact that we can reproduce a body with its proper form but on a different scale” [Hölder 1900, 15, Eng. trans. 23], Hölder seems to believe that this is the result of a construction from more primary concepts rather than a fact that is ‘given within intuition’.

10. See Hölder’s terminological distinction between ‘primary’ concepts [*ursprüngliche Begriffe*] and ‘primitive’ concepts [*Grundbegriffe*].

construction that might be used to explain them. In this case, all the concepts of an axiomatic theory would be 'given', but several concepts that are 'given' in a theory might not be 'given' in a different axiomatic theory.<sup>11</sup> The distinction between given and constructed would thus be reduced to the opposition between primitive and derived, and would of course make sense only inside an axiomatic framework. This distinction is relative (to a given axiomatic theory). Yet, the term 'given' does not occur in this meaning in the paper on measurable magnitudes, where Hölder builds an axiomatic theory: on the contrary, 'given' is used there only in the technical sense of 'data' in a mathematical proof.

3. According to the genetic meaning, a 'given' is something whose properties can be determined without any activity, i.e., without executing any operation. According to this meaning, geometrical concepts are partly 'given' and partly not. This distinction is best developed in *The Mathematical Method*, but is already present in *Intuition and Reasoning*, especially where Hölder makes reference to a paper by Konrad Zindler.

My claim will be that Hölder's use of the term 'given' can be understood according to the third sense both in *Intuition and Reasoning* and in *The Mathematical Method*, where he discussed mathematical questions such as the difference between arithmetic and geometry, or the nature of primitive propositions in mathematics (see sections 2.1.2 and 2.2). The term is on the contrary used in the epistemological meaning when Hölder makes reference to the philosophical discussion about the genesis of geometrical knowledge (see section 3).

### 2.1.2 Given versus constructed

The distinction between given and constructed concepts was influenced, as Hölder himself declared, by an article published in 1889 by a student of Meinong: Konrad Zindler, who claimed that geometry takes representations of space as given and does not investigate their origin.<sup>12</sup> Zindler believed that the problem of the correspondence between geometrical truths and the external world is a question raised by physics that does not concern geometry.<sup>13</sup>

11. See for example Pasch's remarks on geometrical concepts: "These concepts are said to refer to position. They are thus introduced as derived with respect to geometrical fundamental concepts. Yet, if, without adding other geometrical concepts, one derives from them further concepts taken to refer to position, then one might take the derived concepts as stem concepts [*Stammbegriffe*] that precede the others inside the group of concepts that refer to position" [Pasch 1882, 74, my trans.].

12. "In mathematics but not in philosophy of mathematics should the axioms be the starting point" [Zindler 1889, 2ff.].

13. "In mathematics the representation of space should be considered as given. It is unnecessary to take into account how our representation could have originated—geometry is indispensable also in physiological and psychological investigations on the meaning of space and already finds the most extensive application—, how it relates



Although Hölder mentioned Zindler in several parts of *Intuition and Reasoning*, he did not explicitly mention Zindler's distinction between geometrical objects that are taken as given and arithmetical concepts that are not.<sup>14</sup> Yet this division is quite similar to Hölder's own distinction between concepts that are given and concepts that are constructed: in both cases axiomatic concepts are opposed to concepts that occur in the theory of numbers. According to Zindler, in arithmetic there is just one determinate primitive concept from which all others can be considered as developed, whereas in geometry there are several concepts that cannot be inter-defined, such as direction, point, distance, coincidence of figures, area, volume, and space [Zindler 1889, 1–3].<sup>15</sup>

Hölder, like Zindler, believed geometrical primitive concepts to be axiomatic inasmuch as they are taken as given, but unlike him he distinguished two kinds of axiomatic concepts: concepts that denote objects and concepts that denote relations [Hölder 1901, 3n, Eng. trans., I, 247]. Zindler, on the contrary, had claimed that all primitive concepts of geometry denote comparative relations, quoting Meinong's article [Meinong 1882] on the theory of relations for the study of compatibility and incompatibility between relations.<sup>16</sup>

## 2.2 The demarcation between geometry and arithmetic

The distinction between given and constructed is explicitly used in *The Mathematical Method* to demarcate arithmetic from geometry. The difference relies on the fact that arithmetic concerns only constructed, or—as Hölder called them in *The Mathematical Method*—synthetic concepts, i.e., concepts that have their source in some activity, whereas geometry concerns both constructed (synthetic) and given concepts.<sup>17</sup> The examples of given concepts

---

to an objective real space, what sense could this question about an objective real space have and whether this question has any sense at all" [Zindler 1889, 2ff.].

14. Yet, both in the paper on measurable magnitudes and in *The mathematical method* Hölder mentioned Zindler's use of the term axiomatic to denote concepts that occur in the axioms. See [Hölder 1901, 3n, Eng. trans., I, 247], [Hölder 1924, 10] and [Hölder 1924, 10].

15. "When we look for the primitive geometrical representations we cannot specify a single wholly determined primitive concept that might be considered as the starting point of the development of other concepts, but there are here several fundamental representations that are reciprocally irreducible and undefinable. Such are for example the direction, the point, the distance, the concept of coincidence or of superposition of geometrical figures, of surfaces, of volumes, and of course of space itself" [Zindler 1889, 3].

16. "Objects that fall under one of the previously introduced primitive concepts are accessible comparative relations, as the possibility of their quantitative measurement (measure of lengths, measure of surface, measure of solids, measure of angles) shows" [Zindler 1889, 2ff.].

17. Hölder distinguished the synthesis that is at stake in the formation of concepts from the synthesis that is at stake in the formation of judgements and that is as-

mentioned in 1924 include those mentioned in 1899, i.e., point, line and plane, which are concepts denoting objects, but also concepts that denote relations, such as the relation of betweenness. As an example of a concept that is on the contrary synthetic (and precisely hypothetico-synthetic), Hölder mentions a line segment, because it can be defined as a couple of points [Hölder 1924, 10–11]. Another example of a constructed concept is the square, which can be constructed from the concepts of segment and angle. More generally, all geometrical constructed (hypothetico-synthetic) concepts are obtained by executing a series of operations in a given order: the possibility of executing those operations in the mentioned order is guaranteed by the axioms [Hölder 1924, 10–11]. Synthetic concepts, like given concepts, might denote an object or a relation: for example, the square denotes an object, whereas the property of a plane figure of being obtained by a—perpendicular, parallel or central—projection from another figure is a relation [Hölder 1924, 12].

This insistence on the difference between concepts of objects and concepts of relations is important a) to determine the extent of Meinong's influence, b) to explain the tension between Hölder's own conception of geometry and the modern axiomatic conception, and c) to understand how Hölder managed to demarcate arithmetic from geometry: the main difference between the two relies exactly on the fact that the latter contains some concepts that cannot be defined synthetically.<sup>18</sup>

In arithmetic one considers only things that correspond to our activity together with their relations: mathematical concepts are thus all constructed, but not from given concepts, so they are called 'pure synthetic concepts': examples are numbers, but also groups and permutations.<sup>19</sup> In arithmetic there

---

sociated to the distinction between analytic and synthetic propositions. This use of the term 'synthetic' is thus different from Kant's use, as Hölder himself remarked. One would expect Hölder to avoid, for the sake of clarity, to use a Kantian term in a non-Kantian meaning, given that he elsewhere remarked that many philosophical distinctions that were originally introduced with one meaning and are subsequently maintained with a different meaning usually generate more confusion than clarification of concepts [Hölder 1924, 292n, 361]. So, presumably, he must have had some strong reasons for maintaining the Kantian term, as if he wanted his approach to remain Kantian, notwithstanding the differences.

18. "Thereby, between the arithmetico-combinatorial domain and geometry (or respectively mechanics) there is the essential difference that in the former domain one does not assume any such given element as in geometry the point and the line, nor any given and non synthetically definable—i.e., non logically derivable—relation, nor any law (axiom) with respect to those elements and their relations" [Hölder 1924, 295].

19. "With reference to what I have just said, I would like to distinguish the hypothetico-synthetic concept from the pure synthetic concept, depending on the fact that the concept is or is not set up on assumptions similar to those that are made in geometry and in mechanics. One could express it by saying that there is something empirical that inheres in the hypothetico-synthetic concept, so that the elements that compose it together with the relevant relations have some meaning in experience, and also the previously assumed laws (axioms) originate in experience or

are no given objects, so the construction cannot be hypothetico-synthetic, and there cannot be any axioms.<sup>20</sup>

Even if in arithmetic there are no ‘given’ concepts, there are of course primitive concepts that are used to build other concepts, as happens in the arithmetical activities of composing, removing, assembling and separating that generate concepts of higher order, wholes composed of parts [Hölder 1924, 295–296]. Hölder explicitly compared his pure synthetic concepts with Meinong’s objects of higher order, considered as wholes composed of parts and their relations: e.g., a rectangular triangle whose internal surface is green. But Hölder remarked upon a relevant difference: according to Meinong, the parts and the whole derive their meaning from external experience, while this holds, according to Hölder, only for geometrical concepts [Hölder 1924, 296]. Hölder added that Meinong’s distinction between *Zusammenstellung* and *Zusammensetzung* could be interpreted as relying on an opposition between two distinct kinds of operations: on the one hand, to add and to remove, to bring together and to pull apart, to order in a sequence and to assign; on the other hand, to superpose segments, to join bodies, and to apply forces [Hölder 1924, 296–297]. This is interesting, because the operations mentioned with respect to Meinong’s *Zusammenstellung* are the same that occur in the examples of pure synthetic concepts: to order in a sequence is the basic activity for natural numbers taken as positional numbers (*Stellenzeichen*), whereas the relation of assigning is the activity on which the notion of cardinal number (*Anzahl*) is based [Hölder 1924, 161ff.].

## 2.2.1 Hölder and modern axiomatics

Modern axiomatics is based on the idea that different primitives can be used in the axiomatization of the same theory. Furthermore, the properties of the

---

intuition” [Hölder 1924, 295]. The examples are quoted in the introduction [Hölder 1924, 6].

20. “In arithmetic the abstraction of new general concepts and rules thus forms in a way a constituent of deduction, and the procedure providing the basis of abstraction is a process we do not count among experience in a narrower sense of the word, that is to say in opposition to reasoning. Having said this, I want to assert that arithmetic—at least in so far as it concerns whole and rational numbers [...]—has no axioms in the proper sense, and that the arithmetical proof process is not a purely formal process, but a mixed and extremely complex process” [Hölder 1900, 61–62, Eng. trans. 43]. See also [Hölder 1924, 6, 295ff.]. The distinction between arithmetic, that contains definitions but no axioms, and geometry, that contains axioms, was introduced by Hermann Graßmann in a similar way: “Thus proof in the formal sciences does not extend beyond the sphere of thought, but resides purely in the combination of different thought processes. Consequently, the formal sciences cannot begin with axioms, as do the real; rather, definitions comprise their foundation. (Although postulates have been introduced into the formal sciences, for example in arithmetic, this is to be regarded as an error, only to be explained by the corresponding treatment of geometry.)” [Graßmann 1844, 22, Engl. trans. 23].

primitives might be altered and different systems obtained. How are these two things possible, if geometry has to begin from 'given' concepts, implying that these concepts cannot be altered at will?

Hölder's belief that among the given concepts of geometry there are not only relational concepts, such as betweenness, containment and congruence, but also primitive concepts denoting objects, illustrates a relevant difference between Hölder and Hilbert. With this respect, Hölder is nearer to Veronese than to Hilbert, because he considered certain geometrical data as non alterable, as concepts that cannot be different from what they are taken to be in our representation of space. This might be a reason why he did not consider a non-Archimedean theory of magnitudes, even if he had developed a non-Archimedean model of it (see further section 4).

Yet, sometimes Hölder seemed to admit that one might substitute a derived concept for a given one in a system.

As a result of a modification of the whole system of geometry it might anyway happen that afterwards a synthetic concept also occurs in place of a concept that was previously treated as given.  
[Hölder 1924, 12]

Does Hölder mean here that given concepts can also be presented as synthetic concepts, provided the axiomatic formulation of the theory is modified? The above passage seems to introduce a tension between the genetic meaning of given used in the previously mentioned passages, and the axiomatic meaning that seems to be used here. Some remarks on Hölder's conception of the axiomatic method might be helpful to clarify this issue.

The preference for the Euclidean and the Kantian use of deduction and the suspiciousness towards the modern construction of "connected chains of inferences" [Hölder 1924, 2] might be related to the fact that an axiomatic system cannot itself describe the content of axiomatic systems having different axioms. This is one reason why deduction cannot be used to study deduction itself. In particular, symbolical logic cannot establish the relation between different axiom systems and experience. Van der Waerden has suggested a Gödelian interpretation of the above-mentioned passage, claiming that Hölder's mistrust in the possibility of formalizing the whole of mathematics relied on the idea that the concepts and inferences developed at one stage of mathematics immediately underwent a logical analysis (e.g., counting the steps of the deduction), and thereby became themselves objects of investigation [van der Waerden 1938, 161].

This interpretation, although developed in order to show that Hölder had somehow foreseen Gödel's critical results, might also be used as an explanation of the problem mentioned above: how is it possible that concepts based on some activity assume the role of 'given'? Concepts of higher order that are evidently derived by means of inferences, might themselves become objects of investigation at some higher level (that of logical analysis, for example), and thus play the role of 'given': so, for example, something that is obtained by

a first-level activity might itself become a ‘given’ in metamathematical investigations, where the kind of activity involved in the construction of synthetic concepts is of a higher kind.

### 3 The origin of ‘given’: Kantian intuition vs. empirical experience

Paul du Bois-Reymond in the introduction to his *The General Theory of Functions* [du Bois-Reymond 1882, 138] had referred to the debate on the foundations of mathematics as an opposition between two positions: the idealists, admitting the existence of limits, and the empiricists, trying to limit mathematics to the realm of finite experience. Hölder—attesting thus his familiarity (if not affiliation) with the Kantian tradition—suggested that the fundamental question in philosophy of mathematics is the relation between intuition and deduction. On the one hand Hölder discussed the role of intuition in proofs, as I will briefly outline in section 3.1. On the other hand he discussed the role of intuition in the origin of given geometrical concepts: are they derived from *a priori* intuition (Kant) or taken from experience (Helmholtz, but also Lotze, Baumann, Volkelt)?<sup>21</sup> In the first case, intuition is “considered as the source of geometrical knowledge”, whereas in the second case it “appears, conversely, as a result of experience” from which geometrical knowledge is derived [Hölder 1900, 3, Eng. trans. 16]. This question, together with its effects on Hölder’s stance on apriorism and empiricism, will be discussed in section 3.2.

#### 3.1 Intuition and deduction

Otto Hölder’s fundamental aim in *Intuition and Reasoning*, as well as in *The Mathematical Method* was precisely the analysis of mathematical deduction. He claimed that intuition plays no role in deduction, or rather, as Pasch had rather put it in the *Lectures on Modern Geometry* (1882), should not play any role in a complete and reliable deductive process.<sup>22</sup> Mathematical rigor is

21. Here the term ‘given’ is used in the epistemological sense introduced in section 2.1.1.

22. “The fundamental propositions should include all empirical material mathematics has to deal with, so that once those propositions have been set, there is no need to go back to sense perceptions anymore. Actually, in order for geometry to be really deductive, the process of deduction should everywhere be independent from the *sense* of geometrical concepts, and also independent from figures. Only the relations between the geometrical concepts laid down in the used propositions or definitions can be taken into account. On the contrary, it is certainly admissible and useful for deduction, but in no way necessary, to take into account the meaning of the geometrical concepts that occur in it, so that precisely when this becomes necessary, the incompleteness of deduction and, if the gap cannot be filled by some change in the reasoning, the unreliability of the propositions that have been previously used as

Pasch's main aim: intuition should be eliminated from deduction but its presence in the axioms might be accepted, because it is harmless or even precious in order to understand the empirical origin of mathematics. Yet Hölder raises the question once more and in an explicit way, against those philosophers (i.e., Kant) who believe that intuition might play a role in deduction too.

Now, what role does intuition play in deduction itself? Are the elements being combined in the deduction, and the rules according to which they are combined, all what is taken from intuition, or respectively from experience? Or does intuition co-operate also in the single steps of deduction? The latter point of view has been mainly defended by philosophers and seems to have been also Kant's point of view, as he saw in intuition the essential principle for obtaining geometrical knowledge. [Hölder 1900, 11, Eng. trans. 20]

To show that intuition does not necessarily occur in geometrical deduction, Hölder suggests a procedure that differs from the one followed by Pasch:<sup>23</sup> the best way to make Euclid's implicit appeals to intuition explicit, is to transform geometry into a calculus, as Peano did in his *Geometrical Calculus* [Peano 1888].<sup>24</sup> Since a purely logical proof in geometry is possible—even if intuition plays a role in the kind of reasoning that leads effectively to the discovery of a certain result [Hölder 1900, 15, Eng. trans. 23]—it is useless to assume an *a priori* intuition of space in order to explain geometrical proofs (the Kantian

---

demonstration means arise" [Pasch 1882, 98, my trans.]. On Pasch's philosophy of mathematics see in particular [Schlimm 2010].

23. Pasch did not transform geometry into a calculus, but was rather interested in the duality principle and in the possibility of deriving theorems from primitive propositions by logical means rather than by observation of given figures. This was obtained by identifying as primitives several relevant relations between points and lines, so that it could be possible to interpret the primitive propositions by means of alternative concepts (lines and points respectively) and yet obtain valid theorems [Pasch 1882, 98].

24. "By means of such a formulation of all intuitive assumptions, one can divest geometrical deduction itself of intuition. If one denotes the geometrical elements and the operations on these elements through symbols and through successions of symbols—just as in algebra one symbolically represents the numerical magnitudes and the operations to be carried out on them—so, in the simplest cases, it is possible to entirely reduce the geometrical deduction into a calculus that is carried out on symbols, just as in algebra the inferences concerning numerical magnitudes are carried out by means of the so-called letter-calculus. Such a calculus has already been accomplished for single domains of geometry; Peano has assembled these symbolical procedures" [Hölder 1900, 14, Eng. trans. 22]. Hölder is well known for his remarks on the limits of symbolism (see e.g., section 2.2.1 above and especially the contribution by Mircea Radu in this volume [Radu 2013]), but this does not mean that he denied the use of symbolical calculi or its merits in certain cases, as it is evident from the above quotation.

thesis is not necessary for the understanding of deduction).<sup>25</sup> Against the usual objection that geometrical deduction is based on intuition because it makes recourse to auxiliary figures, Hölder argued that this procedure corresponds to the introduction of appropriate existential propositions. To the objection that in certain proofs the geometer reasons by analogy with objects of intuition, Hölder answered that such proofs can be rigorously formulated without any recourse to analogy [Hölder 1900, 10–13, Eng. trans. 20–21].

### 3.2 A partial criticism to Kant

Even if he suggested several arguments in defense of the empirical origin of geometrical objects, in the 1899 lecture Hölder did not attempt to advance a definitive argument against the Kantian interpretation. There is hardly any doubt that Hölder shared an empirical conception of geometry. His remarks on the origin of geometry from raw experience and the subsequent process of idealization<sup>26</sup> do not substantially differ from Pasch's remarks on the origin of geometrical objects.<sup>27</sup> Yet, Hölder did not explicitly adhere to a specific version of empiricism. Although strongly influenced by Helmholtz, as is particularly evident both in the 1901 and in the 1924 essays, Hölder considered two possible ways to explain Pasch's idea that geometrical objects derive from experience: either they derive from immediate perceptions (Helmholtz) or they derive from memories of perceptions (Baumann, Wundt) [Hölder 1900, 3, Eng. trans. 28]. Furthermore, Hölder's arguments offered a general defense of empiricism rather than a defense of Helmholtz's approach.

If Hölder evidently shared an empiricist account of geometry, why was he so cautious about the Kantian conception? Why did he not refute it? There are probably some practical reasons: Hölder was presenting the state of the

---

25. "The recent knowledge of the possibility to represent a geometrical proof in a purely logical way frees us from the necessity to assume an *a priori* intuition of space for the sake of the geometrical proof" [Hölder 1924, 28].

26. "We figured ourselves the geometrical concepts of point, straight line, surface and so on together with their corresponding relations as formed according to experience, certainly a relatively raw experience [...]. Once we arrived at certain regular relations, idealization arose: we imagine that there could be exact points, exact straight lines, and that one might establish exactly whether a given point lies on a given straight line or not, and at the same time we assume that for these exact concepts the regular relations or axioms have unconditioned validity" [Hölder 1924, 397].

27. "The successful application that geometry encounters continuously both in natural sciences and in practical life is based only on the fact that geometrical concepts originally corresponded exactly to empirical objects. Even if they were gradually overdrawn by a net of artificial concepts in order to promote the theoretical development, and inasmuch as one limits oneself, from the beginning, to the empirical nucleus, geometry maintains the character of a natural science, distinguished from other parts of the science of nature, because it needs to derive immediately from experience only a very small amount of concepts and laws" [Pasch 1882, iii, my trans.].

matter concerning the foundational problem of geometry rather than imposing his own point of view with definitive arguments—a reasonable thing to do in an inaugural lecture. This might also explain why he did not defend just one version of empiricism, but discussed the differences between the appeal to immediate perceptions and the explanation based on remembered perceptions. But there might also be methodological and epistemological reasons, as I will claim in the following section.

### 3.2.1 Empiricism is consistent and preferable

Firstly, Hölder considered himself as a working mathematician that investigated the theory of knowledge of his own discipline rather than as a professional philosopher [Hölder 1924, 1], and thus did not venture in a confutation of a philosophical position.<sup>28</sup> He contented himself with arguments showing that empiricism was consistent and preferable. That Hölder held a clear preference for the empiricist viewpoint, is clearly attested by the fact that he put forward some arguments to show that the empirical origin of geometry might be legitimate or even preferred to the idea that the properties of geometrical objects are determined by an *a priori* intuition. The arguments are not presented as a confutation of Kant, but rather aimed at defending the legitimacy, if not the preferability, of alternative conceptions.

Secondly, Hölder refuted the objection that the empirical origin of geometrical concepts might be based on a fallacy of circularity: as Helmholtz had shown,<sup>29</sup> there are some observations—facts of immediate perception—“that are not themselves grounded in geometrical considerations” [Hölder 1900, 4, Eng. trans. 17]. For example, one might derive the geometrical notion of a straight line from some physical observations concerning the line of sight or a tensioned thread, and the distinction between a plastic and a rigid body from the observation that after manipulation of a given material, it might be more or less easy to recover the original visual representation of the object [Hölder 1900, 5–6, Eng. trans. 16–17]. Hölder also considered the fact that the properties of rigid bodies might be determined by a set of subsequent measurements and the geometrical theorems might be the result of induction, but he acknowledged that one cannot thereby exclude the possibility that the Kantian notion of a *a priori* intuition might be compatible with that explanation. Yet,

---

28. Like Zindler, Hölder believed that to investigate geometry from a mathematical point of view, it suffices to know that some of its concepts are taken as given: it is not necessary to discuss how they are given to us.

29. See in particular *The facts of perception* [Helmholtz 1884, 218], but also *Die Anwendbarkeit der Axiome auf die physische Welt* [*The applicability of the axioms to the physical world*], supplement to *Die neuere Entwicklung von Faraday's Ideen über Elektrizität* [*The recent development of Faraday's ideas about electricity*] [Helmholtz 1884, 405]. Note that the title itself of Hölder's lecture is inspired by Helmholtz's question: “Was ist Wahrheit in unserem Anschauen und Denken? [What is truth in our intuition and reasoning?]” [Helmholtz 1884, 218].



in favour of the empirical interpretation, he claimed that the Kantian recourse to a general notion of *a priori* intuition

- is more “artificial”,
- is not strictly necessary to solve the problem, and in fact it is combined with some empirical understanding of the phenomena, and
- does not give an equally detailed explanation.<sup>30</sup>

Hölder admitted that the Kantian conception could not be refuted, but he was confident that future results in the history of mathematics, psychology and physiology might furnish further arguments in favor of the empirical interpretation. The history of mathematics might prove helpful, because a clarification of the origin of geometrical concepts in the Egyptian culture might prove that they originally had a practical aim. The physiological analysis of the spatial properties of objects might suggest that immediate perceptions need to be decomposed, so that there cannot be a unique *a priori* intuition but different forms of intuition depending on vision, tact, muscular sensations, and movement [Hölder 1900, 7, Eng. trans. 18]. This section anticipated some remarks that can be found in the fourth chapter of Poincaré’s *Science and Hypothesis* [Poincaré 1902], but which were already in the essay on *Geometry and Space* [Poincaré 1895].<sup>31</sup> Rather, Hölder’s remarks are inspired by Helmholtz’s writings and Klaus Volkelt’s *Experience and Reasoning* [Volkelt 1886].<sup>32</sup>

A further argument advanced by Hölder was a reasoning to the best explanation: empiricism can better explain human’s capacity to localize an object in a determinate point of space. Quoting Lotze [Lotze 1884, 547ff.], Hölder argued that the reference to individual perceptions can better explain the human capacity to locate an object in a singular point of space.<sup>33</sup> A general *a priori* intuition that is assumed as a necessary condition of mathematical

---

30. “Certainly, a strict follower of Kant would still suspect that any of those results obtained through measurement were already influenced by ready and exact intuition taken as a condition of any experience. Maybe, if he wants to hold on to his point of view, he could not be refuted. On the other hand, it cannot be denied that his standpoint is artificial, and it seems legitimate to drop that standpoint, if one believes one can do without it” [Hölder 1900, 6–7, Eng. trans. 18].

31. Hölder did not quote Poincaré in 1899, but quoted *Science and Hypothesis* [Poincaré 1902] in 1924, as he discussed indirect proofs, the analysis of induction, the compatibility of non-Euclidean geometries with experience, and the distinction between the visual, the tactile and the motor space. It can thus be reasonably assumed that he had not read Poincaré’s essay *Space and Geometry* [Poincaré 1895] before giving the 1899 Inaugural Lecture.

32. In this text, Volkelt developed a theory of knowledge that inherited some features from Descartes, Kant and Locke. He claimed that the starting point of knowledge had to be subjective; yet objective—or rather trans-subjective—knowledge could be reached once experiences were combined with logical deduction.

33. “The importance of the localization problem will certainly be denied by those who consider the pure intuition of space as a condition of any perception. But they must certainly admit that the empirical part of my perception, i.e., also my sensory perceptions, must contain something that allows me, in a single case, to locate an object in this way, and not otherwise, namely to see the object in this determinate

judgement either cannot do that or has to be interpreted as merely introducing an order between the empirical data, "but then we would basically be back to the empirical point of view" [Hölder 1900, 17n, Eng. trans. 32].

In 1899, Hölder believed that it cannot be definitively established that geometrical intuition derives from experience, because an *a priori* intuition might also explain our knowledge of spatial objects. Yet, deduction is not grounded in intuition, so, whatever the philosophical debate on the origin of geometrical concepts, it should have no relevance for the study of mathematical deduction [Hölder 1900, 8, Eng. trans. 19].

### 3.2.2 Kant's thesis can neither be proved nor refuted

Secondly, Hölder considered Kantian geometrical apriorism as a hypothesis that is non provable and non refutable, and as such has poor explanatory power. In *The Mathematical Method* Hölder advanced the same argument in defense of empiricism that he had already suggested in *Intuition and Reasoning* (see above, section 3.2.1): the Kantian objection to Helmholtz's theory does not hold because there is no fallacy of vicious circle. But then he added:

I would only like [...] to remark here that this [Kantian] conception seems to be neither strictly provable nor strictly refutable, and not even directly fruitful, because of its abstract nature.  
[Hölder 1924, 368]

The same remark is repeated in § 132, where Hölder discusses criticism towards Kantian apriorism: the only way to defend *a priori* intuition is to assume a platonic idea of space as a non provable hypothesis.<sup>34</sup> Hölder's answer to this rhetorical question had already been given in *Intuition and Reasoning*: the Kantian hypothesis does not have any explanatory power, because it gets rid—too easily—of several relevant problems.

It seems to me that the empiricist view already has an advantage in the fact that it allows a detailed explanation of intuition, while the hypothesis assumed by the follower of Kant cuts off all the rest. Indeed, it is always an advantage to set up new problems.  
[Hölder 1900, 6–7, Eng. trans. 18]

---

position and not in some other position with respect to my body (cf. the theory of 'local signs' by Lotze [...])" [Hölder 1900, 16n, Eng. trans. 31].

34. "Under the given circumstances it seems to me that one can hold the Kantian assumption only if one assumes, so to speak, behind the representative space an *a priori* intuition that operates in it but is incompletely expressed by it, a sort of platonic idea of space that is the *a priori* form and the condition of all experience. Such an assumption could hardly be refuted. Actually, it is a hypothesis that is not supported by any evidence. [...] And does it truly explain the applicability of mathematics to experience?" [Hölder 1924, 384–385].

### 3.2.3 The Kantian epistemological project is still viable and fruitful

Thirdly, even if he did not accept the role attributed by Kant to intuition in geometrical judgements, Hölder still considered Kant's fundamental question "how is pure mathematics possible" as a fruitful guideline for the epistemological and logical investigation of mathematics [Hölder 1924, 3]. In *The Mathematical Method*, Hölder was involved in a truly epistemological project and characterized it as an answer to Kant's question:

– "How is pure mathematics possible?",

which can be divided, in the case of geometry, into two different questions:

1. How can geometry be deductively built from its premises? and
2. Where do the axioms come from?

The two problems identified by Hölder in 1899 and analyzed with respect to the different role of intuition—the origin of geometrical concepts and the investigation of mathematical deduction—are here presented as two aspects of Kant's foundational program. Furthermore, Hölder's terminology is even more Kantian, because the expression 'synthetic' is here preferred to 'constructed'.

### 3.2.4 Kant was right about arithmetic

Finally, Hölder agreed with Kant's solution as far as arithmetic is concerned. In particular, Hölder preferred Kantian apriorism to the extreme consequences of an empiricist point of view, as for example the application of psychologism to arithmetic, and the abandonment of the distinction between *a priori* and *a posteriori* knowledge.

The acceptance of an empirical understanding of the origin of geometrical axioms does not imply—according to Hölder—any change in the Kantian and Platonic understanding of arithmetic as a necessary and unconditioned kind of knowledge.<sup>35</sup> For this reason, Hölder did not share the position of the empiricists who claim that arithmetic also has an empirical origin:

I am very far from speaking in favor of an extended empiricism, for example a psychological conception that might transform the laws of reasoning into habits of reasoning. [Hölder 1924, 7]

And for the same reason Hölder defended the distinction between *a priori* and *a posteriori* knowledge.<sup>36</sup>

---

35. "That arithmetic is an *a priori* science is certainly assumed by most mathematicians. [...] The point of view that Plato, and among the moderns especially Kant, validated, i.e., that there is unconditioned necessary knowledge that is grounded in what is for us condition and criterion of all knowledge, might well remain in place. But the borderline between what is necessary (*a priori*) and what belongs to experience is dragged to a different position" [Hölder 1924, 7].

36. "On the other hand, it is certainly possible to deny an unconditioned distinction between two kinds of knowledge, an *a priori* and a *a posteriori* kind of knowledge. Yet, even if one does it, one cannot consider all reasoning processes as equivalent" [Hölder 1924, 395].

## 4 The case of measurable magnitudes

The previous remarks about Hölder's conception of geometry and 'given' concepts are necessary to understand Hölder's approach to the theory of measurement, because the latter is taken to be a science that has axioms, like geometry or mechanics [Hölder 1901, 3, Eng. trans. I, 237]. The axioms express three properties of order—trichotomy, density, and Dedekind's continuity—and three properties of addition—positivity, solvability and associativity. If the theory of magnitudes contains axioms about 'given' concepts, why is Dedekind's continuity included among them?

In the following, I will try to understand why Hölder considered Dedekind's continuity an indispensable property of measurable magnitudes, and what role was played in the choice of the axioms for continuity both by the philosophical understanding of geometry delineated in section 2 and by some new mathematical results by Hölder himself. This might help us in understanding why he included such a strong axiom as Dedekind's postulate—which can be derived from the axiom of Archimedes, together with the other axioms for measurable magnitudes—withstanding his knowledge of Veronese's non-Archimedean theory and his own development of an original non-Archimedean model.

### 4.1 Theory of measurement and geometry

Like geometry and mechanics, the theory of measurement is also based—according to Hölder—on axioms.

It is a different matter with geometry and mechanics, where certain axioms based upon sensory experience (or, as some want it, upon intuition) are presupposed. As with geometry and mechanics, the theory of measurable magnitudes can be based upon a set of facts which I will call "axioms of magnitude" or "axioms of quantity." [Hölder 1901, Eng. trans. I, 237]

Hölder did not give any argument to prove the analogy. The latter cannot be based on the fact that the theory of magnitudes shares all axioms with geometry, because Hölder claimed that the axioms of measurement do not occur as such in geometry. Even if the axioms of the theory of measurement hold for the comparison and the addition of straight lines and surfaces, they can be applied to geometry only provided certain further axioms are also assumed: "Special axioms must be formulated again for geometrical applications" [Hölder 1901, Eng. trans. II, 346]. For example, the axioms of the theory of measurable quantities can be applied to equalities between figures, only if the following fact is also assumed, that two lines can be compared, and be either equal or unequal.

On the contrary, the axioms of the theory of measurable magnitudes satisfied by line segments follow from purely geometrical

axioms for points and segments on a line. [Hölder 1901, Eng. trans. I, 237]

In particular, the different assumptions that need to be made in the case of lines and in the case of plane figures explain the difference between the equality of lines, that is taken as ‘primary’, and the equality of content of figures, that is taken as ‘constructible’.<sup>37</sup> Notwithstanding these differences, the relationship between geometry and theory of measurement is yet so close, that Hölder claimed their relative consistency.<sup>38</sup> The strict relationship to geometry is also a reason why Hölder’s axiomatic system was based on the operation of addition.<sup>39</sup>

## 4.2 The continuity axiom

According to Hölder, the origin of the axioms of the theory of measurement is similar to the genesis of the axioms of geometry: they are all derived from experience. We are thus justified in believing they will not lead to contradiction. This cannot of course be proved by experience, but the empirical genesis and the empirical fruitfulness in applications are considered by Hölder as reliable reasons for the choice to include an axiom into the system (see section 2.2 above and section 4.3 below).

When Hölder published his essay on measurement, there were several other texts discussing the same topic: two of them, written by Giuseppe Veronese and Rodolfo Bettazzi did not include the postulate of Dedekind among the set of axioms that characterize measurable magnitudes. More precisely, neither of them included the axiom of Archimedes, nor, *a fortiori*, the postulate of Dedekind, which can be derived from the axiom of Archimedes together with the other axioms of the theory of magnitudes.<sup>40</sup> Hölder was well acquainted

---

37. “While with line segments it must be assumed that they can be compared and necessarily found to be either equal or unequal, one can prove from axioms, in which the word “area” does not occur, and on the basis of certain definitions, that two figures can be compared with respect to area [...]. This is based on the fact that equality of line segments is a primary concept and equality of areas in figures is a derived geometrical concept [...]” [Hölder 1901, 3, Eng. trans. modif. I, 247].

38. “From the consistency of the axioms of magnitudes one can derive the consistency of the geometrical axioms ( $\alpha$ ) to ( $\kappa$ ) in Part II of this work and vice versa” [Hölder 1901, Eng. trans. I, 248n].

39. “Geometrical and physical magnitudes, too, such as line segments, areas, times, masses, forces, etc., as given in the first place, can only be added, and addition is only possible with magnitudes of the same kind. Talk of the multiplication of geometrical or physical magnitudes is based upon arbitrary convention. The present paper assumes only such axioms as relate to the comparison and addition of magnitudes. These axioms are sufficient to deduce the Euclidean theory of proportion and the contemporary theory of measurement. The results concerning the multiplication of magnitudes appear here only as corollaries to one of these theories” [Hölder 1901, Eng. trans. I, 250n].

40. The axiom of Archimedes states that given two quantities  $a$  and  $b$  such that  $a < b$ , there exists a natural number  $m \in \mathbb{N}$  such that  $ma > b$ . The postulate of

with both essays—he quoted the latter and discussed extensively the former—as well as with the Euclidean theory of proportions, which he declared to be the original nucleus of the theory of measurable magnitudes, containing the axiom of Archimedes as the general form of continuity. So, Hölder must have had some good reasons to include in a theory that declares to have an empirical origin the postulate of Dedekind that cannot be verified empirically, and that was not included in the ancient theory of proportions presented in Euclid's *Elements* nor in recent theories of measurement. What were his reasons?

The choice of axioms is up to a certain point arbitrary,<sup>41</sup> claimed Hölder, yet he had reasons to include Dedekind's axiom rather than alternative formulations of continuity. Why did he not content himself with the axiom of Archimedes, which is sufficient to compare any two magnitudes? Hölder declared that his approach differed from those of Veronese and Bettazzi, but did not explain what was wrong with their proposals. I suggest that he did not share Veronese's and Bettazzi's choice of the axioms because he did not share their epistemological aims, or at least what he took them to be. Hölder interpreted the theory of Veronese as absolutely incompatible with Dedekind's theory of continuum, and the notion of magnitude developed by Bettazzi as a mathematical formulation of the properties of special kinds of abstract magnitudes rather than a theory that could apply to the usual physical and geometrical magnitudes.

I intend only to propose a simple system of axioms from which the properties of the ordinary continuum of magnitudes can be derived; I do not intend to establish special kinds of magnitudes as was done by Bettazzi [...]. [Hölder 1901, Eng. trans. I, 247n]

Taken as such, Hölder's argument claims that one should hold to what is usually considered to be a magnitude, and maintain the standard notion of continuity. Hölder's reasoning seems to be an argument *ad populum*: let's conform to what the majority of mathematicians usually do, i.e., to the standard notion of continuity, which was expressed by the Archimedes axiom in Euclid's time but which is now usually formulated along Dedekind's lines. If this was the argument, then it would not be so stringent. A brief comparison with Bettazzi, Veronese and Dedekind might help us to clarify Hölder's point of view.

---

Dedekind asserts that if a system of quantities can be divided in two parts  $A$  and  $B$  so that each element of  $A$  is at the left of each element of  $B$ , that is to say, that each element of  $A$  is strictly inferior to each element of  $B$ , then there is one and only one element that determines this section: it will be the maximum of  $A$  or the minimum of  $B$ .

41. "Of course, the axioms chosen are arbitrary, up to a point, and it is a matter of convenience whether Dedekind's axiom of continuity plus axioms I to VI or some other axioms are to be preferred" [Hölder 1901, Eng. trans. I, 248n].

### 4.3 Bettazzi's and Veronese's approaches

What are the mistakes and the ambiguities that Hölder criticized in recent theories? First of all, Dedekind's postulate was presented as an arithmetical axiom,<sup>42</sup> but according to Hölder the principle used in arithmetic is not an axiom but a definition, whose consistency has to be proved.<sup>43</sup>

Secondly, a theory of measurement has the principal aim of accounting for physical operations executed on empirical objects: the operation of addition should be in this sense more fundamental than the relation of order. Yet, in the arithmetical analyses of the continuum, the latter is often based on some order relation rather than on addition. But what was wrong with Bettazzi's and Veronese's conceptions? Before answering this question, I will briefly summarize the approach developed by Bettazzi and Veronese, who partly shared Hölder's epistemological aims, but arrived at different conclusions. This will help us to understand that Hölder's choice not to follow them was based on epistemological grounds.

In Bettazzi's *Theory of Magnitudes* [Bettazzi 1890] the properties of order were introduced after the properties of addition, and continuity was taken to be a property of magnitudes and not of numbers. Notwithstanding these similarities, Bettazzi followed an abstract approach based on the construction of structures that did not have an empirical counterpart—a method that Hölder could not accept because it contrasted with his own empirical epistemology. According to Bettazzi's definition, a class of magnitudes was an Abelian additive semigroup, and could be defined independently from order [Ehrlich 2006], [Cantù 2010]. Different kinds of order relation were taken into account and different abstract classes of measurable magnitudes were introduced: additivity—which suffices, together with a relation of equality or inequality, to determine the notion of a magnitude—was introduced independently from a total order relation, which was considered as necessary to introduce measurement. The order relation was not defined from the outset as Dedekind-continuous: on the contrary, different kinds of order allowed measurement, which was defined not only for one-dimensional classes but also for multidimensional classes that can be decomposed into ordered subclasses: in particular, non-Archimedean classes. So, magnitudes were primarily defined as a structure and not by reference to some physical operation that allows us to measure things.

---

42. On the contrary, in the theory of quantities Hölder gives a geometric formulation of the postulate. "This last axiom is the axiom of continuity in geometrical form. Dedekind first noted that the nature of the continuity of straight lines can be expressed by this principle [Dedekind 1872, 93]. At the same time, Dedekind put less value upon postulating this axiom for geometry, rather he took it as an opportunity for his definition of the irrational numbers" [Hölder 1901, Eng. trans. II, 354].

43. "If one considers that the definition of each step requires a specific law, so the concept of the totality of all steps means that we believe we can conceive the totality of all the laws, which nonetheless require a certain stipulation. Yet, such a wholly indeterminate totality could represent an inadmissible concept [Hölder 1924, 193–194].

In Veronese's *Foundations of Geometry* [Veronese 1891], continuity was also taken to be a property of magnitudes rather than of numbers, and geometry was taken to be a mixed science, whose postulates are partly empirical and partly mathematical extensions of empirical facts. This point of view does not seem too far from Hölder's own epistemological approach: the postulate of continuity is exactly one of these 'ideal' or purely logical concepts that do not have a counterpart in experience, because it involves the concept of limit. In particular, Veronese believed that Dedekind's continuity, being defined for numbers and being a part of arithmetic, didn't need to be the only possible or adequate description of the empirical world. He thus developed a different notion of continuum, discussed in detail by Hölder himself.

Veronese's continuum can be roughly described as a weaker form of continuity that can be geometrically represented by a sheaf of straight lines and that includes Dedekind's continuity as a special case. Each straight line is Dedekind-continuous, and thus also Archimedean, but the whole sheaf, ordered from bottom to top and from left to right, is not [Cantù 1999, 107ff.]. Veronese's system is thus a generalization of a Dedekind-continuous subsystem and includes a generalization of the axiom of Archimedes: "for any two magnitudes of the system such that one is smaller than the other, there is a multiple of the smaller that exceeds the bigger". Only, the notion of multiple is not the usual one, based on natural numbers, but is built on an enlarged set that includes infinitely small and great numbers.

Hölder, like most of his contemporaries—with the exception of Hans Hahn [Hahn 1907] and Luitzen Egbertus Jan Brouwer [Brouwer 1907]—overlooked this aspect of Veronese's theory, but focused only on the fact that Veronese, like Bettazzi, questioned whether the Archimedean axiom might be a necessary feature of magnitudes, and of measurable magnitudes in particular.

#### 4.4 Hölder's reasons to include Dedekind's continuity

Hölder admitted that it cannot be verified by our experience that Dedekind's continuity is an indispensable property of measurable magnitudes, but he considered this assumption as justified on the basis of some empirical experiences and applications.

I will have to explain later (§ 99) that the question which one of two concepts 'precedes' the other cannot always be definitely answered. But in this case, to defend the Dedekindian axiom of continuity, it would suffice to point out the fact that an axiom gains importance in mathematics by allowing us, together with the other introduced axioms, to dominate a multifaceted factual domain. I will furnish evidence of this in the case of the axiom of continuity; for example, by means of this axiom, significantly



extended to time intervals, I will prove that of any two points moving in the same direction on a straight line, the one that happens to have a bigger velocity must overtake the other. [Hölder 1924, 88]

Hölder thus advanced pragmatic reasons for the acceptance of Dedekind's continuity as a property of measurable magnitudes. But why could he not admit a more general and weaker property, allowing Dedekind's continuity to be, as in the case of Veronese, just a special case of the former? The relevance for the pragmatic applications could have thus been maintained. I suggest that the answer is related both to some significant technical results to be found in Hölder's 1901 paper and to Hölder's preference for an empirical conception of geometry discussed in section 3.

#### 4.4.1 Two technical results

As Philip Ehrlich has shown, there are two important results to be found in Hölder's paper on measurable quantities that concern non-Archimedean continuity: 1) the proof that the axiom of Archimedes can be derived from the postulate of Dedekind's cuts, together with the other axioms of Hölder's theory of measurable magnitudes, and 2) the construction of a particular model of a non-Archimedean continuum.

- (1) Hölder showed that the axiom of Archimedes can be derived from the postulate of Dedekind together with the other axioms of measurable magnitudes [Hölder 1901, 10ff., Eng. trans. 248]. Hölder was not the first to prove this result, but he had several predecessors: Pasch, Veronese, Stolz, Bettazzi, Hilbert. Pasch [Pasch 1882, 125–126], in particular, was praised by Hölder as the one who first derived the axiom of Archimedes from a projective formulation of Dedekind's continuity postulate [Hölder 1924, 89]. Hölder's proof was already advanced in 1899, and thus before Hilbert's book on the foundations of geometry, even if it is usually quoted from the 1901 paper [Ehrlich 2006, 59–61].
- (2) Hölder himself had developed an original model of a non-Archimedean continuum that differs from those introduced by Veronese, Bettazzi and Hilbert.<sup>44</sup> Ehrlich has shown that Hölder's model of a non-Archimedean continuum differs from the one by Veronese, because it satisfies Veronese's Principles I–IV but not Veronese's Principle V, which expresses the generalized version of the Archimedes axiom and allows us to consider Dedekind's continuity as a special case of the non-Archimedean continuum

---

44. Whereas Hölder discussed in detail the model by Veronese, which is also based on a geometrical understanding of magnitudes, he did not extensively analyze Bettazzi's model, which appears to him as too abstract, nor Hilbert's model, which was based on arithmetical functions.

[Ehrlich 2006, 97].<sup>45</sup> Unlike Veronese's model and like Du Bois-Reymond's theory, the system developed by Hölder satisfies divisibility [Ehrlich 2006, 94], which is relevant in physical measurements, because it guarantees the divisibility of a magnitude in submultiples.

Hölder rightly assumed the counternominal implication of his result on the derivability of the Archimedes axiom from Dedekind's continuity. He thus held that if the Archimedes axiom  $A$  can be derived from the Dedekind's postulate  $D$  and the others axioms of measurable magnitudes  $\Gamma$ , then, if  $A$  is negated,  $D$  and  $\Gamma$  cannot both hold.<sup>46</sup> So, if one wants Dedekind's continuity to hold for pragmatic reasons, then one cannot refute the Archimedean axiom.

Now, given the original results developed by Hölder, and in particular the fact that he had built a new non-Archimedean model, why did he stick to the Archimedean property in the axiomatic formulation of the theory of measurable magnitudes? I will suggest that this technical result played a fundamental role. A main reason why Hölder assumed from the beginning the validity of the Archimedean axiom, even if he had constructed a consistent non-Archimedean model, was that he believed that in refuting it, he would have to abandon Dedekind's continuity altogether. This is true in Hölder's non-Archimedean model, but not generally in Veronese's model, for Dedekind's continuity can be saved as a special case. Yet, why did he overlook the relation between Veronese's continuity and Dedekind's continuity? My argument is that his epistemological framework and its difference from Veronese's perspective might play a role here.

#### 4.4.2 The epistemological framework

The mathematical choice of different axioms in the case of Hölder and Veronese should be related to different epistemological frameworks. Both Veronese and Hölder took inspiration from Pasch, but Veronese was interested in a mathematical hypothesis concerning an order relation that might differ from the standard one, and yet be compatible with our intuition of physical objects (a bundle of parallel sticks, for example). Hölder's approach was on the contrary the result of an exclusive interest in the empirical foundation of measurement.

Veronese considered geometry as a mixed science (partly *empirical* and partly *ideal*, i.e., going beyond the empirical realm), or—as Hölder would have said—the result of a combination of *hypothetico-synthetic* concepts and *pure synthetic* concepts. Hölder, on the contrary, considered all primitive propositions of geometry—and thus of measurement theory—as axioms, i.e., as based

---

45. Veronese's absolute continuity also known as 'Archimedes generalized' is formulated as follows: "if  $a$  and  $b$  are two intervals of the system and  $a < b$ , there is a determined symbol of multiplication (number)  $\eta$  such that  $\eta a > b$ ". Hölder's property of divisibility requires that "for each member  $A$  of the system and each positive integer  $n$ , there is an  $X$  in the system such that  $nX = A$ ".

46. If  $(\Gamma \wedge D) \vdash A$ , then  $\neg A \vdash \neg(\Gamma \wedge D)$ .

on experience. According to him, the hypotheses that Veronese was trying to anchor to experience, did not really apply to experience—and in fact the model that Hölder suggested as a realization of Veronese's system, was based on functions and not on magnitudes [Hölder 1901, 12n, Eng. trans. 249].<sup>47</sup> On the contrary, the Archimedean property, apart from being present in Euclid's *Elements* (Book VI, § 9), appeared to him as relevant for empirical applications and as a standard component of our "usual continuum" [Hölder 1901, 11, Eng. trans. 249]. Dedekind's continuity was praised for its compatibility with the Archimedean axiom but also for its important role in proofs and applications. But there is one more reason: using Dedekind's continuity, it is possible to

establish at the same time a very tight connection between the Euclidean theory of proportions and the the notion of measurement on the one hand, and the arithmetical theory of irrational numbers on the other hand. [Hölder 1901, 19, Eng. trans. 241]

Hölder assumed the definition of the arithmetical continuum, and then showed that the modern theory of proportion (or theory of measurement) can be grounded only if one shows not only that to any geometrical measure of a magnitude there corresponds a number, but that also to any geometrical sum of two magnitudes  $a$  and  $b$  there corresponds the arithmetical sum of the corresponding numbers that measure  $a$  and  $b$ . He thus introduced, like Bettazzi before him but unlike Veronese, a representation theorem that justifies the fact that we can apply numbers to measure magnitudes. Veronese, on the contrary, did not conceive the idea of a representation theorem, because he was far away from the idea of establishing a tight connection between arithmetic and geometry. On the contrary, he aimed to show that the notion of continuity is different in geometry and in arithmetic, and that, given the fact that the former is fundamental, the latter might not be an adequate description of continuity in general.

The radical demarcation between arithmetic and geometry is not only a matter of epistemological preference or philosophical inclination: it affects, and not in a regressive way, Hölder's mathematical works, because it is at the basis of the representation theorem that he developed in *The Axioms of Quantity*. Having assumed that real numbers are a symbolical representation of empirical magnitudes, Hölder perceived the necessity to prove that, if the symbolic system should measure (represent) the system of magnitudes, it should have a similar structure. This was not due to the fact that he had overlooked the mathematical investigation of new systems of quantities: on the contrary, he developed an original model of non-Archimedean magnitudes. Rather, having related the theory of quantities to geometry and the latter to certain empirical operations of superposition, Hölder saw no reason for adopting a higher generality in his approach to measurement.

---

47. That they could not be falsified by experience itself was not a sufficient argument, because neither Veronese's continuum nor Dedekind's continuum can be.

## 5 Conclusion

I will now summarize some of the complex and interesting relations between Hölder and other mathematicians and geometers of the 19th century, as they emerge from the reconstruction of Hölder's epistemology suggested in this paper. Hölder's relations to previous or contemporary authors are not linear, but complex and multifaceted: this explains the originality of his position, which could be *prima facie* assimilated to that of other authors, but reveals, upon deeper investigation, its peculiarity.

### 5.1 The demarcation between geometry and arithmetic

In section 2, I have investigated the distinction between 'given' and 'constructed' that Hölder had introduced already in *Intuition and reasoning*, although he clarified it further in *The Mathematical Method*: rather than being just a difference concerning the origin of mathematical concepts (as was the case in Gauß, who declared geometry to be an empirical science), it is a difference concerning the possibility of defining the concept by means of a constructive activity. It cannot be reduced to the modern axiomatic distinction between primitive and derived concepts, nor to the epistemological distinction between *a priori* and *a posteriori*, but is rather related to the formal genesis of the object itself.

While talking of activity, Hölder said something similar to what Hermann Graßmann had described as the genesis of pure mathematical concepts through laws. The same does not hold for geometry, according to Graßmann, because the latter is not a pure mathematical theory but has to assume the three-dimensionality of space as something given. Even if Hölder had a different understanding of what is given in geometry, because he considered points and lines among given concepts, he had a similar understanding of what given is opposed to, i.e., something that can be defined by its own constructive law. Hölder's use of the distinction given-constructed to demarcate geometry from arithmetic is quite near to Graßmann's use of the distinction between what is posited by thought and what stands in front of our thought, a distinction that Graßmann used to demarcate formal from real sciences. But there are also relevant differences, because Hölder remarked that not all geometrical concepts are given. The mixed nature of geometry is more similar to the conception of Giuseppe Veronese, who had made an effort to reconcile the belief in the empirical genesis of basic geometrical truths with the ideal constructions required by non-Euclidean, non-Archimedean or hyperspace geometries.

Anyway, Hölder did not quote Graßmann, but Zindler, a disciple of Meinong who insisted on the fact that there is only one primitive concept in arithmetic while there are many different concepts in geometry: for this reason geometrical primitives are mainly concepts denoting relations. On this

point Hölder differed from Zindler, but also from Veronese, Hilbert and modern axiomatics. Against the tendency introduced by Pasch to identify geometrical primitives as specific relations between elements (e.g., betweenness), Hölder insisted on the necessity of distinguishing two kinds of axiomatic concepts: concepts denoting objects and concepts denoting relations. This remark can be read as a reaction against the tendency to call geometry any abstract theory on relations of betweenness, connection, order that did not take into account certain fundamental properties of geometrical objects. With this respect, Hölder's position is again similar to that of Veronese, who refused to admit that certain mathematical theories, like Poincaré's theory of the hyperboloid of one sheet or Hilbert's non-arguesian theory, could be called 'geometry', exactly because they did not preserve certain properties of the geometrical object "straight line".

The demarcation between arithmetic and geometry was further developed in *The Mathematical Method*, where a different terminology was used for constructed concepts, being now called hypothetico-synthetic (in geometry) and pure synthetic (in arithmetic). Furthermore, the distinction is compared with Meinong's opposition between *Zusammenstellung* and *Zusammensetzung*, resulting in a clarification of the different kinds of activities that can be respectively associated to arithmetic (to order in a sequence or to assign) and geometry (to superpose segments). Hölder adhered to Kant's apriorism as far as arithmetic is concerned, defending the distinction between *a priori* and *a posteriori* reasoning processes.

## 5.2 Axiomatic method and mathematical practice

Pasch's effort to eliminate intuition from deduction was the starting point of most mathematicians and logicians of the time. Hölder was highly sensitive to this issue and was willing to adopt Peano's symbolic reconstructions of geometry as a calculus in order to show that intuition does not play any role in deduction. The latter was in effect the center of Hölder's interest, but in a quite original and different way with respect to the contemporary dominant traditions in axiomatics.

Hölder's interest was directed to the formulation of the primitive propositions of a given theory, and to the analysis of the kind of activities involved in deduction itself rather than to the investigation of independence and consistency results. Although sharing with Couturat and Russell a certain conception of arithmetic as purely formal, Hölder held a radically different conception of logic: it is not a universal theory that precedes all mathematical disciplines, but a disciplinary investigation aiming at isolating the specific deductive patterns that are used in different branches of mathematics.

From this perspective, he was much more interested in the analysis of increasing levels of complexities in our mathematical activities, and in the limits of the use of deduction to study deduction itself. For this very reason, he

adopted a different approach, based on a case by case investigation of mathematical deductive procedures that cannot itself be developed into a symbolic language, but can be accomplished only by means of the natural language.

This idea of a general logic that arises from a never definitively achieved investigation of mathematical deductive activities explains the distance with respect to Hilbert's effort to prove metamathematical results inside axiomatic systems, but also the radical difference with respect to Peano, who aimed to express all mathematics by means of a unique symbolic language, or even adopting a standardized universal language like the *latino sine flexione* for scientific communication. Hölder on the contrary, believed that the specificity of mathematics had nothing to do with the symbols used to represent it, but rather with the kind of activity, i.e., deduction, deployed in its practice. For this reason he developed a sort of empirical study of deduction: this is one of the most original aspects of Hölder's investigation, and probably also a reason why his main book went largely unnoticed in his time.

### 5.3 Philosophy of mathematics

The interest in the investigation of mathematical deduction and in the construction of a general logic that can be derived from the examination of specific cases is the core of Hölder's epistemology of mathematics. Yet, in *Intuition and Reasoning*, and again in *The Mathematical Method* he often discussed the position of different philosophers on the questions of apriorism and empiricism. Hölder's point of view is clear: such questions belong to the philosophy of mathematics but do not have any influence on how mathematical deduction works or is developed. The mentioned questions can be asked with respect to the genesis of mathematical objects, but do not have anything to do with what mathematics actually is.

This might be a reason why Hölder's position was partially unresolved, both with respect to empiricism and Kantian apriorism. Another reason might be the fact that Hölder was not only a mathematician, but had truly philosophical and argumentative skills that urged him to distinguish his own preferences or inclination for a philosophical position from the possibility of corroborating or confuting its claims.

The adhesion to the Kantian project was not evident in *Intuition and Reasoning*, neither in the terminology—no use of the term “synthetic” was made—nor in the conception of geometry, yet Hölder apparently refrained from a too strong and explicit criticism of Kant's apriorism. On the contrary, in *The Mathematical Method* there is an explicit revival of the Kantian epistemological project, which is mentioned in the introduction as the starting point of the whole essay. Besides, Hölder's conception of arithmetic is compatible with the Kantian perspective, even if the criticism towards the Kantian conception of geometry is more explicit than in his inaugural lecture.

A similar ambivalent relation was developed by Hölder towards empiricism. On the one hand, he was strongly influenced by Pasch concerning the foundation of geometry, and by Helmholtz concerning the theory of measurement. Notwithstanding the general acceptance of a genetic theory of geometrical objects and measurable magnitudes, Hölder resisted some possible consequences of an empiricist position. For example, he abhorred the application of psychologism to arithmetic, and the extension of empiricism to ‘pure’ mathematics.

## 5.4 Conclusive remarks

The difficulty of classifying Hölder as well as his ambivalence towards apriorism and empiricism are already a proof of the unconventionality of his position. His ability to analyze advantages and limits of the axiomatic method and of the traditional debate in the philosophy of mathematics is typical of a scholar who was both a practicing mathematician and a practicing philosopher, and believed in the inexhaustibility of both practices. It also reveals a deep knowledge of the contemporary mathematical and philosophical debate, but unfortunately only up until the First World War, because *The Mathematical Method* appeared in 1924 but in an almost unrevised version with respect to its initial project.

Hölder’s epistemological approach does not prove to be generally regressive, if one considers the relation to his own mathematical research: sometimes the effect is innovative, as is the case for the distinction between arithmetic and geometry that allowed him to develop a representation theorem, even if it was sometimes limitative, as in the case of the strict association between geometry and measurement, that made the development of a higher generality useless, or pragmatically uninteresting.

## Bibliography

BETTAZZI, RODOLFO

1890 *Teoria delle grandezze [Theory of Magnitudes]*, Pisa: Spoerri.

BROUWER, LUITZEN EGBERTUS JAN

1907 *Over de grondslagen der wiskunde. Akademisch proefschrift*, Amsterdam: Maas & van Schutelen, Eng. trans. “On the Foundation of Mathematics”, in *Collected Works*, North-Holland, Amsterdam, 1975, vol. 1, 13–101.

CANTÙ, PAOLA

1999 *Giuseppe Veronese e i fondamenti della geometria*, Milano: Unicopli.

2010 The role of epistemological models in Veronese’s and Bettazzi’s theory of magnitudes, in *New Trends in Logic and Philosophy of*

*Science*, edited by D'AGOSTINO, M., GIORELLO, G., LAUDISA, F., PIEVANI, T., & SINIGAGLIA, C., London: College Publications, 229–241.

DARRIGOL, OLIVIER

2003 Number and measure: Hermann von Helmholtz at the crossroads of mathematics, physics, and psychology, *Studies in History and Philosophy of Science*, 34(3), 515–573.

DEDEKIND, RICHARD

1872 *Stetigkeit und irrationale Zahlen*, Braunschweig: Vieweg & Sohn, Eng. trans. by W. W. Beman and revised by W. Ewald “Continuity and irrational numbers”, in W. Ewald (ed.), *From Kant to Hilbert*, vol. 2. Oxford, Clarendon Press, 1996, 765–778.

DIEZ, JOSE

1997 Hundred years of numbers. An historical introduction to measurement theory 1887–1990. Part I: The formation period. Two lines of research: Axiomatics and real morphisms, scales and invariance, *Studies in History and Philosophy of Science*, 21(1), 167–181.

DU BOIS-REYMOND, PAUL

1882 *Die Allgemeine Functionentheorie [The General Theory of Functions]*, Tübingen: Laupp.

EHRlich, PHILIP

2006 The Emergence of non-Archimedean *Grössensysteme*. The Rise of non-Archimedean Mathematics and the Roots of a Misconception, *Archive for History of Exact Sciences*, 60(1), 1–121.

GAUSS, CARL FRIEDRICH

1900 *Werke*, Leipzig: Teubner.

GRASSMANN, HERMANN

1844 *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik, dargestellt und durch Anwendungen auf die übrigen Zweige der Mathematik, wie auch die Statistik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert*, Leipzig: Wigand.

GRAY, JEREMY

1994 Otto Hölder and group theory, *The Mathematical Intelligencer*, 16(3), 59–61.

HAHN, HANS

1907 Über die Nichtarchimedischen Grössensysteme [On non-Archimedean Systems of Quantities], *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften Wien: Mathematisch-Naturwissenschaftlichen Klasse IIa*, 116, 601–655.



HELMHOLTZ, HERMANN VON

1884 *Vorträge und Reden*, vol. 2, Braunschweig: Vieweg.

HILBERT, DAVID

1899 Grundlagen der Geometrie, in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmal in Göttingen*, Leipzig: Teubner.

HÖLDER, OTTO

1900 *Anschauung und Denken in der Geometrie. Akademische Antrittsvorlesung gehalten am 22. Juli 1899*, Leipzig: Teubner, Eng. trans. *Intuition and Reasoning in Geometry* by P. Cantù and O. Schlaudt, this volume, 15–52.

1901 Die Axiome der Quantität und die Lehre vom Mass, *Berichten der mathematisch-physischen Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig*, 53, 1–64, Eng. trans. J. Michell and C. Ernst, “The Axioms of Quantity and the Theory of Measurement”, *Journal of Mathematical Psychology*, 40, 232–252, 1996.

1924 *Die mathematische Methode [The Mathematical Method]*, Berlin: Springer.

1926 Der angebliche circulus vitiosus und die sogenannte Grundlagenkrise in der Analysis [The alleged vitiosus circle and the so-called foundational crisis in analysis], *Ber. sächs. Ges. Wiss. Leipzig*, 78, 243–250.

1929 Der indirekte Beweis in der Mathematik [The indirect proof in mathematics], *Ber. sächs. Ges. Wiss. Leipzig*, 81, 201–216.

1930 Nachtrag zu meinem Aufsatz über den indirekten Beweis [Supplement to my article on the indirect proof], *Ber. sächs. Ges. Wiss. Leipzig*, 82, 97–104.

1931 Axiome, empirische Gesetze und mathematische Konstruktionen, *Scientia*, 49, 317–326, French trans. *Axiomes, lois empiriques et constructions mathématiques* by H. Buriot-Darsiles and Moulins-s-Allier, *Scientia*, Supplément, 49, 133–141.

LAMBERT, JOHANNES H.

1764 *Neues Organon oder Gedanken über die Erforschung und Bezeichnung des Wahren und dessen Unterscheidung vom Irrthum und Schein [New Organon]*, Leipzig, repr. Akademie-Verlag, Berlin 1990.

LOTZE, HERMANN

1884 *System der Philosophie. Zweiter Theil: Metaphysik. Drei Bücher der Ontologie, Kosmologie und Psychologie*, Leipzig: S. Hirzel, 2nd ed.

MEINONG, ALEXIUS

1882 Zur Relationstheorie [On the theory of relations], *Sitzungsberichte*

*der phil.-hist. Classe d. K. Akademie der Wissenschaften zu Wien*, (101), 573–752.

MICHELL, JOEL

1993 The origins of the representational theory of measurement: Helmholtz, Hölder and Russell, *Studies in History and Philosophy of Science*, 24, 185–206.

PASCH, MORITZ

1882 *Vorlesungen über neuere Geometrie [Lectures on Modern Geometry]*, Leipzig: Teubner.

PEANO, GIUSEPPE

1888 *Calcolo geometrico secondo l'Ausdehnungslehre di H. Graßmann*, Torino: Bocca, Eng. trans. by L.C. Kannenberg, *Geometric Calculus: According to the Ausdehnungslehre of H. Graßmann*, Boston: Birkhäuser, 2000.

POINCARÉ, HENRI

1895 L'espace et la géométrie [Space and Geometry], *Revue de Métaphysique et de Morale*, 3(6), 631–646.

1902 *La Science et l'Hypothèse*, Paris: Flammarion, Eng. trans., *Science and Hypothesis*, London: Walter Scott Publishing, 1905.

RADU, MIRCEA

2003 A debate about the axiomatization of arithmetic: Otto Hölder against Robert Graßmann, *Historia Mathematica*, 30, 341–377.

2013 Otto Hölder's interpretation of David Hilbert's axiomatic method, *Philosophia Scientiæ*, 17(1), 117–129.

SCHLIMM, DIRK

2008 On abstraction and the importance of asking the right research questions: Could Jordan have proved the Jordan-Hölder theorem?, *Erkenntnis*, 68(3), 409–420.

2010 Pasch's philosophy of mathematics, *The Review of Symbolic Logic*, 3(1), 93–118.

VAN DER WAERDEN, BARTEL L.

1938 Otto Hölder. Nachruf [Obituary], gesprochen in der öffentlichen Sitzung vom 25. Juni 1938, *Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig*, 90, 97–101.

VERONESE, GIUSEPPE

1891 *I fondamenti della geometria [Foundations of Geometry]*, Padova: Tipografia del Seminario.

VOLKELT, JOHANNES

1886 *Erfahrung und Denken: kritische Grundlegung der Erkenntnistheorie [Experience and Reasoning]*, Hamburg: Voss.

ZINDLER, KONRAD

1889 Beiträge zur Theorie der mathematischen Erkenntnis [Contributions to the theory of mathematical knowledge], *Sitzungsberichte der phil.-hist. Classe d. K. Akademie der Wissenschaften zu Wien*, 118(9), 1–98.