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**The Epistemological Thought of Otto Hölder**

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# **Otto Hölder's Interpretation of David Hilbert's Axiomatic Method**

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# Otto Hölder's Interpretation of David Hilbert's Axiomatic Method \*

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**Résumé :** L'article présente une reconstruction brève de la conception de preuve développée par Otto Hölder. La reconstruction se concentre sur la critique de la conception axiomatique de David Hilbert en général et sur sa conception des métamathématiques en particulier. On affirme que l'analyse faite par Hölder des idées méthodologiques générales de Hilbert et surtout de la structure logique des preuves fournie par Hilbert dans les *Grundlagen der Geometrie* (1899) est très utile pour comprendre plus clairement la thèse de van der Waerden affirmant le lien entre la conception de la preuve développée par Hölder et la tradition établie par Kurt Gödel.

**Abstract:** In this paper I provide a brief reconstruction of Otto Hölder's conception of proof. My reconstruction focuses on Hölder's critical assessment of David Hilbert's account of axiomatics in general, and of Hilbert's conception of metamathematics in particular. I argue that Hölder's analysis of Hilbert's general methodological ideas and, more importantly, Hölder's analysis of the logical structure of the proofs provided by Hilbert in his *Grundlagen der Geometrie* of 1899 are helpful in reaching a clearer understanding of van der Waerden's claim linking Hölder's conception of proof to the tradition established by Kurt Gödel.

## 1 Otto Hölder's forgotten legacy

Hölder's reputation as a mathematician is well established.<sup>1</sup> Things look different with respect to Hölder's ideas about the philosophy of mathematics.<sup>2</sup>

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\* I thank the anonymous reviewers for their remarks that led to important improvements of an earlier version of this paper.

1. Compare [Waerden 1939].

2. These ideas can be found in a series of works published between 1892 and 1935.

The most important objective of Hölder's philosophical thinking is the clarification of the value and of the limitations of sign-use in mathematics [Hölder 1924, 5]. Hölder is concerned about the emergence of a *Zeitgeist* characterized by a unilateral confidence in symbolic practices, which he sometimes calls "formalism".

The first clear expression of this concern can be found in Hölder's critique of Robert Graßmann's conception of rigor [Hölder 1892]. Graßmann claims that rigor can only be reached by putting in place a symbolic practice in which the fundamental unity of concept and symbolic expression, between *analysis* and *synthesis*, is conveyed. We recognize here Leibniz's ideal of developing a thinking calculus, a *Calculus Ratiocinator*.<sup>3</sup> In his 1892 review of Robert Graßmann's book, Hölder tries to establish that Graßmann's approach falls short of accomplishing the methodological ideal of putting up a calculus expressing the full harmony between conceptual content and symbolic expression. Hölder's arguments also suggest, more generally, that the shortcomings of Graßmann's approach prove that Leibniz's methodological ideal pursued by Graßmann is utopian.<sup>4</sup>

Hölder's concern about the best way to understand the relationship between conceptual content and symbolic expression in mathematics was emphasized by van der Waerden. In his *Nachruf auf Otto Hölder*, which was published in 1939, not long after Hölder's death, van der Waerden praises Hölder as someone having significantly contributed to an important transformation of mathematics "from the formal to the critical, from mere computation to concept" [Waerden 1939, 55, my translation].<sup>5</sup> And van der Waerden goes on to claim that Kurt Gödel's methods and results were the confirmation of Hölder's conception of method. Unfortunately, van der Waerden's remarks are brief, so that the precise nature of the link between Hölder's account of method on the one hand and Gödel's methods and his philosophy of mathematics, on the other, is left open.

The discrepancy between van der Waerden's suggestion of a link between Hölder and Gödel, on the one hand, and the lack of recognition of Hölder's philosophy of mathematics by posterity, on the other, is surprising. If van der Waerden is right, modern history and philosophy of mathematics has failed

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3. One important consequence of this is that, according to Graßmann, a rigorous treatment of mathematics must be given independently of any theory of logical inference. For Graßmann, a rigorous treatment of logic is only possible if logic is developed as a symbolic calculus, that is, as a chapter of mathematics. Hölder agrees with the former claim, but dismisses the latter (see below).

4. Here, I am, of course, only proposing an interpretation of Hölder's critique of Graßmann. In the 1892 review of Graßmann's work, Hölder does not explicitly mention this aspect of Leibniz's conception. He mentions it, however, in 1924, in a discussion of the place and value of symbolic methods in developing the logic of relations. Hölder calls Leibniz's idea of a *Characteristica Universalis* inadequate [Hölder 1924, 274].

5. Hölder died in August 1937. Van der Waerden's *Nachruf* was received by the *Annalen* in May 1938. It was published in 1939.

to take notice of an immensely fruitful contribution, provided by a first rate mathematician. How can this be explained?

Gödel's results concerning the limitations of a formal-axiomatic treatment of arithmetic were published in 1931, seven years after the publication of Hölder's *The Mathematical Method* [*Die mathematische Methode*]. As far as I can tell, Gödel nowhere mentions Hölder's work. In the present context, the interesting question is rather, how did Hölder receive Gödel's results? This major event could not have escaped Hölder's attention. So far, however, I have been unable to find any documents recording Hölder's reaction to Gödel's results.<sup>6</sup>

Some of the remarks made by Hölder in 1924 concerning the work of Gottlob Frege and of Giuseppe Peano on the foundations of arithmetic provide an interesting hint. Hölder emphasizes that the foundations of arithmetic cannot be secured through a symbolic calculus. Hölder thinks that a careful analysis of the symbolic proofs found in writings such as those of Frege and Peano *would* reveal a vicious circle: it would turn out that these proofs are based on the same *sequential procedures*, which they are supposed to justify [Hölder 1924, 349].<sup>7</sup>

Hölder's rejection of the methods of Frege and Peano and, more generally, his strict rejection of symbolic logic (see below) may be one of the causes having led to the poor reception of his ideas.

The structure and style of Hölder's approach to these issues in his writings may be the other cause. Hölder does not manage to develop a sufficiently clear and rich philosophical conception capable to support his examination of method. He writes not as a philosopher, but as a mathematician interested in a clarification of the mathematical method. This may be seen as a major limitation of Hölder's approach. This limitation, however, is also an advantage, for Hölder does not try to defend any particular philosophical school, nor does he try to create a philosophical system of his own. He rather observes the mathematical practice and the way it is reflected by philosophers, trying to expose the issues on which the philosophical description fails to do justice to the practice of the working mathematician.

The main objective of this paper is to give a brief presentation of some of Hölder's reflections concerning the limits of formalization and the role of *arithmetical intuition* in mathematical deduction. I rely mainly but not exclusively on Hölder's *The Mathematical Method* of 1924.

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6. If I am not mistaken, a detailed comparison between the philosophical ideas of Hölder and those of Gödel is still missing.

7. Arguments of a similar kind are discussed below.

## 2 The limits of formalization

The first important remarks on the subject are made in the *Introduction* to the previously mentioned book. Hölder describes mathematics as a deductive science. Mathematical deduction, we are told, cannot be adequately described in terms of syllogistic logic. An adequate description of mathematical deduction requires the logic of relations.<sup>8</sup> And then Hölder warns against looking for a deductive presentation of logic. Since logic is the science of deduction, it would be inappropriate to seek a deductive theory of deduction [Hölder 1924, 2].

The same topic is taken up a few lines further with respect to the logic of relations. Hölder explains that in the hands of mathematicians, the logic of relations was turned into a symbolic calculus known under the name of “Logistic” [Hölder 1924, 4–5]. Hölder points out that in certain cases formalization may prove helpful but only as a heuristic tool. He continues by rejecting the view (which he attributes to Louis Couturat), that the formalization of logic is an essential requirement for providing a truly scientific presentation of logic. He explains his position as follows:

(...) when in the course of a mathematical investigation new signs (...) are introduced, as this often happens, it is not possible to justify the reasoning behind the introduction of those new signs, by means of a new symbolic calculus. It is obvious that if this could be done, and if the previous position would be adequate,<sup>9</sup> one would then be compelled to justify the latter symbolic calculus by means of a new such calculus, and we would have to pursue this process further indefinitely. We would thus reach (...) an “infinite regress”.<sup>10</sup> [Hölder 1924, 5]

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8. Remarks such as these are typical for Hölder’s time. There is nothing original or surprising about them. This also holds for much of Hölder’s detailed discussion of the limits of syllogistic logic and of the virtues of the logic of relations given in the 12<sup>th</sup> chapter of his book. The 12<sup>th</sup> chapter has the title “Preliminary general logical remarks” [Hölder 1924, 247–277]. The chapter is, of course, important for a reconstruction of the precise terms in which Hölder views this subject. Such a reconstruction, however, goes beyond the purposes of the present paper.

9. Hölder returns to Couturat’s position later in the book. He criticizes Couturat for regarding the symbolic treatment of logic as a necessary condition for logical completeness. Hölder writes: “One must take into account the fact that each (...) sign requires a rule describing its use; this rule, however, cannot be explained by means of a new symbolic expression, but rather based on the meaning [*Bedeutung*] of the sign alone” [Hölder 1924, 177]. In a footnote to this comment Hölder also mentions Brouwer’s paper “Intuitionism and Formalism” [Brouwer 1913]. Hölder emphasizes that from his own point of view, when it comes to the foundations of mathematics, formalism is an inappropriate conception.

10. All the translations of Hölder’s texts are mine.

Hölder relies here on a classical kind of argument often raised to emphasize the limits of justification. One equally classical solution to this type of problem is provided by the axiomatic method.

If we desire to provide an axiomatic treatment of a theory, then all we need to do is to identify a finite adequate list of undefined primitive notions, and a finite adequate list of first principles expressing properties of the primitive notions (the axioms of the theory). Based on the chosen primitive notions and axioms we can then state and deduce all other results of the theory in case. One of the major hopes linked to the new axiomatic method, as it was understood, for instance by David Hilbert and his school, was precisely the conviction that a symbolic, axiomatic treatment of pure mathematics and of logic would make it possible to prove the consistency of mathematics.

Hölder had carefully reflected on these matters independently of Hilbert, and indeed before Hilbert's work on the foundations of geometry was published. As said, the first traces of Hölder's thinking are visible in Hölder's review of Robert Graßmann's *Treatise on Number* [Hölder 1892]. Here we find the first expression of Hölder's conviction that the axiomatic method cannot be used for proving the consistency of arithmetic.

Before returning to Hölder's arguments concerning the nature of mathematical deduction, it is helpful to emphasize an important aspect of Hilbert's conception of proof, criticized by Hölder. Niebergall and Schirn describe Hilbert's conception of proof in the following terms:

While nowadays metamathematics is conceived as being formulated and axiomatized in a formal language, Hilbert's finitist metamathematics qua contentual theory of formalized proof (...) is neither formalized nor axiomatized. It is supposed to allow only intuitive and contentual reasoning, and, as Hilbert (...) stresses, this kind of reasoning is, by its very nature, free from axiomatic assumptions. Due to this sharp distinction between formalized mathematics and contentual metamathematics, the meanings attached to the word "to prove" in the two "disciplines" are fundamentally different. In formalized mathematics, it means *to infer according to the formal rules of the calculus*; in contentual metamathematics, it means *to show by means of contentual intuitively evident inference*. It is precisely the intuition-based character of metamathematical reasoning that is supposed to guarantee its security and reliability. (...) In a nutshell: Hilbert's finitist metamathematics is a non-formalized, non-axiomatized, and surveyable "theory" with a non-mathematical vocabulary whose intended domain consists of concrete and surveyable figures, in particular, of formulas (e.g., axioms) and proofs. [Niebergall & Schirn 1998, 274ff.]

Hölder believes that the difficulties involved in establishing the consistency of formalized mathematics, cannot be conclusively dealt with, by means of the

metamathematical shift proposed by Hilbert. Hölder thinks that the same difficulties would sooner or later surface again in examining the proofs elaborated at the level of contentual metamathematics. We would then, if the ideal of a formal-axiomatic treatment is retained, sooner or later have to mathematize metamathematics. In this case, however, we would run precisely into the infinite regress described by Hölder above.

In the next sections of this paper I will present some of Hölder's key arguments advanced in support of this position. These arguments belong to an alternative conception of mathematical proof outlined by Hölder in his *The Mathematical Method* of 1924.

### 3 A genetic interpretation of Hilbert's axiomatic method

Hölder is not just skeptical about the possibility of proving the consistency of arithmetic along the lines proposed by Hilbert. He finds no merit whatsoever in seeking an axiomatic treatment of arithmetic. The main reason for this is Hölder's understanding of the value and limits of the axiomatic approach in pure mathematics. The fundamental ingredient of this attitude is Hölder's account of the content/form distinction in mathematics. Let me explain this.

An axiomatic approach is needed and indeed natural in those sciences in which there is a clear separation between the *content* dealt with and the *conceptual apparatus* used in describing that content, that is, the theoretical *form* used to shape the content. This is the case in empirical sciences such as geometry or mechanics [Hölder 1924, 465–466]. Hölder's approach to this matter is typical for the positivist approach to natural sciences. In natural sciences the axioms are hypothetical statements about the properties of sensible objects [Hölder 1924, 374–493].<sup>11</sup>

The situation changes considerably if we shift our attention from empirical science to pure mathematics and particularly to arithmetic and logic. In arithmetic and logic it is impossible to draw a clear distinction between the content of the theory and its form.

In order to establish this, Hölder explores a large number of examples. One such example is taken from geometry. We are told that in geometry we often find geometric concepts and geometric proofs, which essentially rely on our logical intuition of the sequence of the natural numbers. In such cases, geometric concepts are used “not just formally but according to their content” [Hölder 1899, 58–59]. Here is one of Hölder's examples used to illustrate this situation.

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11. Hölder quotes, for instance, the views of Moriz Schlick [Hölder 1924, 374–397] and of Ernst Mach [Hölder 1924, 42–45; 464ff.]. I cannot discuss the details of Hölder's conception of the foundations of natural science here.

Take, for instance, Hölder's discussion of the construction of the concepts *square* and *n-gon*. According to Hölder, the definition of the concept *square* is a specification of a finite sequence construction-steps.<sup>12</sup> Finite constructions of this kind do not involve arithmetical intuition. The situation changes in the case of general concepts such as the *n-gon* concept. A definition of this concept must provide a description of a sequential construction procedure. In the case of the *n-gon* concept, however, the construction involved requires an arbitrary, unspecified number of steps. It is therefore impossible to list all the steps required one by one or to actually carry out the construction.

We can easily prove that in any equilateral triangle, square, regular pentagon, etc., the sum of the distances of any interior point to its sides is independent of the position of the point, without using arithmetical intuition. To state and prove a theorem such as: "In any regular *n-gon* the sum of the distances of an interior point to its sides does not depend on the position of the point" we need to go beyond the finite. The idea is that our ability to frame certain general concepts and general proofs must be rooted in a special faculty of our mind, the arithmetical intuition, our intuition of the sequence of the natural numbers [Hölder 1924, 338ff.].

The simplest mathematical expression of arithmetical intuition is the arithmetic of the natural numbers. The core of this is our ability to frame infinite sequential procedures and use them in developing new concepts and proofs. *The sequence-concept [Reihenfolgebegriff] is the content and, at the same time, the form of our thinking* [Hölder 1924, 292–320].

In *Über den Zahlbegriff*, which was published in 1900, Hilbert contrasted the *genetic* to the *axiomatic* approach to arithmetic and remarked that priority should be given to the axiomatic method. Hölder is not against the axiomatic method, but simply against giving it priority over the genetic method (which Hölder prefers to call the *synthetic* method) in respect to consistency questions.

In a section with the heading "The so-called axiomatic presentation of arithmetic illuminated from the perspective of the superposition of concepts [*Die sogenannte axiomatische Arithmetik, beleuchtet vom Standpunkt der Überbauung der Begriffe*]", Hölder gives a clearer shape to his fundamental objection to the "formalism" which he attributes, for instance, to Couturat and Hilbert.<sup>13</sup> Hölder achieves this by investigating Hilbert's proof, given in his *Grundlagen der Geometrie* [Hilbert 1899, 26 ff., 72ff.] of the commutative law for the multiplication of certain objects.

Hilbert presents a system of axioms for "complex systems of numbers [*komplexe Zahlensysteme*]" [Hilbert 1899, 26], a term already used by Hermann Hankel. Hilbert defines his *Zahlensysteme* by introducing

12. First construct a line segment. Choose one of its endpoints. Draw a perpendicular line to the segment through the point chosen, etc.

13. In this context it is perhaps interesting to note that Hölder also criticizes Hermann Hankel's use of the *permanence principle* in similar terms [Hölder 1924, 210–211].

three groups of axioms. They are: “Sätze der Verknüpfung”; “Sätze der Anordnung”; Archimede’s axiom. Hilbert uses these to define a totally ordered, Archimedean field.

Hölder’s general critique of Hilbert’s position is, as said, directed against Hilbert’s belief that the consistency of arithmetic can be proved independently of genetic considerations. At the same time, Hilbert’s 1899 proof of the commutative law of multiplication discussed by Hölder has nothing to do with the consistency of arithmetic. Hölder is aware of it. He obviously believes, however, that his analysis of Hilbert’s proof of the commutative law reveals that Hilbert’s axiomatic treatment of the matter essentially involves contentual arithmetic, or, in other words, that Hilbert’s axiomatic treatment of the complex system of magnitudes, tacitly relies on the genetic approach which it is supposed to avoid. Hölder assumes that any axiomatic treatment of arithmetic will require proofs similar to those discussed by him, so that his arguments would count for those proofs as well.<sup>14</sup>

Hölder begins his examination of Hilbert’s proof by reproducing the main principles involved in his discussion of the proof:

$$a + (b + c) = (a + b) + c \quad (7)$$

$$a + b = b + a \quad (8)$$

$$a(bc) = (ab)c \quad (9)$$

$$a(b + c) = ab + ac \quad (10)$$

$$(a + b)c = ac + bc \quad (11)$$

$$ab = ba. \quad (12)$$

He describes Hilbert’s approach in the following terms:

We imagine certain objects of an arbitrary kind, which are not numbers, but which allow two operations [*Kompositionsweisen*] to be carried out on them. We call the operations “addition” and “multiplication” (...). It goes without saying that these operations must be distinguished from the addition and multiplication of numbers. The two operations are taken to satisfy formulas (1) to (5) above. A few other obvious principles are (...) also stipulated. Of particular importance is the *Archimedean axiom* which is also adopted as a principle. The latter states that, provided that  $a$  is smaller than  $b$ , by adding  $a$  to itself sufficiently often, an object  $a + a + a + \dots + a$  can be manufactured [*hergestellt*], which is greater than  $b$ , that is, there is a multiple of  $a$  exceeding  $b$ . [Hölder 1924, 320ff.]

14. For similar discussions of other proofs compare [Hölder 1924, 313ff.; 503–510; 511–515].

Hölder then reproduces the most important steps of Hilbert's proof. Here I only discuss those steps of Hilbert's proof, which according to Hölder involve arithmetical intuition. To establish (6), Hilbert first proves the equality  $ndmd = mnd$ . Hilbert's proof goes like this.

Hilbert postulates the existence of a special unit  $e$  for which  $ae = ea = a$  holds for any magnitude  $a$ . By repeatedly using axiom (4) we get:

$$a(e + e + \dots + e) = a + a + \dots + a; \quad (10)$$

and by using axiom (5) we get:

$$(e + e + \dots + e)a = a + a + \dots + a. \quad (11)$$

This leads immediately to:

$$a(e + e + \dots + e) = (e + e + \dots + e)a. \quad (12)$$

The proof of  $ndmd = mnd$  is completed as follows:<sup>15</sup>

$$\begin{aligned} Md \cdot nd &= \\ \{(e + e + \dots + e)d\}\{[e + e + \dots + e]d\} &= \\ \{(e + e + \dots + e)d[e + e + \dots + e]\}d &= \\ \{(e + e + \dots + e)\{d[e + e + \dots + e]\}\}d &= \\ \{(e + e + \dots + e)\{[e + e + \dots + e]d\}\}d &= \\ \{\{(e + e + \dots + e)[e + e + \dots + e]\}d\}d &= \\ \{(e + e + \dots + e)[e + e + \dots + e]\}\{d^2\} &= \\ mnd^2. \end{aligned}$$

This step of Hilbert's proof is analyzed by Hölder in the following terms:

This may leave the impression that in this proof proper arithmetical reasoning has been switched off [*ausgeschaltet*]. A closer examination of the genesis [*Entstehung*] of (10) and (11) leads, however, to the contrary view. For instance, in order to obtain (10) by using rule (4), one must first acknowledge that the repeated application of rule (4) to the expression:

$$a(e + e + \dots + e) \quad (13)$$

should be understood, for instance, as meaning

$$a(\dots(((e + e) + e) + e) + \dots + e).$$

The latter operation must produce the object  $a$  as its result, as many times as the object  $e$  was present in the bracket of (13). One therefore finds oneself counting the steps of the proof procedure [*Man denkt sich also wieder die Schritte des Beweisverfahrens selbst gezählt*]. [Hölder 1924, 323ff.]

15. The variables  $m$  and  $n$  stand for natural numbers. The letters  $a$  and  $d$  stand for arbitrary magnitudes.

Examining the previous proof, one may be perhaps inclined to think that it requires an explicit use of mathematical induction or, at least, that explicit induction might be used to rewrite some of the previous steps. But this would lead to adding the induction-axiom to the axioms actually used by Hilbert. Hilbert, however, does not explicitly require induction here and neither does Hölder.<sup>16</sup> Hölder simply emphasizes that this way of using the signs involves counting the steps of the proof while proving. Expressions such as (13) may look finite because the number of iterations in (13) is supposed to be finite. At the same time, however, (13) involves, so to speak, second-order-counting. There is, so to speak, first-order-counting involved in each of the individual steps  $e; (e + e); (e + e) + e$ , etc. However, formulas such as (13) involve more than that. In using them we quit the initial counting process, and view it as accomplished. We proceed by counting the counting steps themselves, thus reaching what I have called second-order-counting or perhaps, even better, meta-counting. In this way certain steps of the proof are turned, while proving, into objects of the proof, which proceeds by further counting former steps of the proof. Therefore each proof of this kind is, not simply a proof about operations on magnitudes but rather also a proof about our way of conceptualizing our operations carried out while proving. This is what Hölder means when he speaks of contentual use of arithmetic and by concept-superposition [*Überbauung der Begriffe*].<sup>17</sup>

Let me now turn to Hölder's comments on Hilbert's use of the Archimedean axiom. Hilbert's proof also relies on the following identities:

$$md < a \leq (m + 1)d \quad (8)$$

and

$$nd < b \leq (n + 1)d. \quad (9)$$

Hölder writes:

An exact analysis of the thinking pattern that led to the inequalities (8) and (9) also leads to a notable result. Using the Archimedean axiom only states that not all multiples of  $d$  can be

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16. Hölder does discuss the place of the induction-axiom in mathematics and argues that inductive proofs require the contentual [*inhaltlich*] use of arithmetical intuition. He does this independently of his reconstruction of Hilbert's proof discussed here [Hölder 1924, 313ff.]. For a very good discussion of the distinction between *formal* and *contentual* induction in Hilbert's work and of the difficulties linked with Hilbert's concept of finite consistency-proofs, compare [Tapp & Lück 2003]. Tapp and Lück consider Poincaré's critique of Hilbert and some of Skolem's ideas on the debate, but they do not mention Hölder.

17. Hölder explores a wide range of examples of this kind, independently of Hilbert's conception, in his detailed discussion of arithmetic [Hölder 1924, 161–198]. In particular, he claims that the commutative law for the multiplication of the natural numbers already involves what I have called second order counting and thus involves concept-superposition.

smaller than  $a$ . From this one deduces that in the sequence of multiples

$$1d; 2d; 3d; 4d; \dots$$

there must be a first that is not smaller than  $a$ . But since  $d$  itself, that is, the first element of the sequence, was taken to be smaller than  $a$ , there must be a multiple of that first element of the sequence for which  $md$  is still smaller than  $a$ , whereas is  $(m+1)d$  greater or equal to  $a$ ; this proves (8). [Hölder 1924, 324]

Here arithmetical intuition is involved twice. First of all, it is involved in the axiom itself. This axiom is nothing more than the expression of the idea that it is possible to add  $d$  an indefinite number of times, until we find some natural number  $k$  satisfying  $a \leq kd$ . The axiom is used, so to speak, to cover up arithmetical intuition. The second time it is involved in establishing the existence of the number  $m$ . To find  $m$  we must conceive another sequential procedure. We pursue the sequence  $1d, 2d, 3d, \dots, kd$  backwards an indefinite number of times until we find the least natural number (which we denote by  $m+1$ ) satisfying  $a \leq (m+1)d$ . Once found, it is clear that  $md < a$  also holds. In order to establish the existence of  $m+1$  we must appeal to the fact that the set of the natural numbers is well-ordered. The Archimedean axiom and the well-ordering principle depend on our ability to imagine certain sequential procedures as being repeated for an unspecified number of times, and therefore, on contentual arithmetic. The proof described here also involves concept-superposition [*Überbauung*] of concepts: in a first step, we use a sequential procedure to generate  $k$ ; this is followed by a second sequential procedure which is attached on top of the first, and which leads to  $m$ .

Hölder concludes his discussion of Hilbert's proof with the following interesting general statement. If the consistency of arithmetic can only be proved axiomatically, as Hilbert desires, we would then have to prove the consistency of the system of axioms chosen. If such a proof were to be given independently of genetic considerations, we would have to rely on yet another system of axioms in order to achieve our goal. This however would require a new system of axioms. We would thus get an infinite regress [*recursus ad infinitum*] [Hölder 1924, 325], so that a complete consistency proof could never be accomplished. It could not be accomplished because, in the course of such a proof, we would be forced to define new infinite procedures and based on them introduce new *synthetic* concepts [*synthetische Begriffe*] in the sense explained above. The Archimedean axiom is, as far as Hölder is concerned, not an axiom but, since it depends on contentual arithmetic, a definition.<sup>18</sup>

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18. Hölder also points out that E. Study and L. E. J. Brouwer share similar views [Hölder 1924, 325, footnotes 1 and 3].

## Conclusion

In the previous pages I presented a small sample of Hölder's reflections concerning mathematical deduction. The most important aspect found is Hölder's view that certain proofs involve what I have called second-order-counting or meta-counting, something that Hölder calls concept-superposition. According to Hölder, therefore, a deductive examination of certain theorems of arithmetic involves self-referential moments. A sharp separation between deduction and counting, of the kind involved in Hilbert's distinction between mathematics and metamathematics appears therefore, impossible.

Niebergall and Schirn point out that there is a complementarity between Hilbert's treatment of arithmetic, which requires coding numbers by signs on the one hand, and the post-Gödelian tradition which relies on coding signs by numbers, on the other [Niebergall & Schirn 1998, 174]. Hölder recognized that counting cannot be switched off, that is, that formalization would require, in a second step, mathematization. In this sense, Hölder's investigations of proof do anticipate Gödel's methods.

Hölder's methodological reflections have a great deal more to offer. I think that Hölder's methodological remarks belong to the intuitionist tradition. His views are compatible with most of the seven tenets of intuitionist thinking proposed by Tieszen [Tieszen 2005, 228ff.]. A detailed reconstruction of Hölder's particular brand of intuitionism is still missing.

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