



Philosophia Scientiæ

Travaux d'histoire et de philosophie des sciences

17-1 | 2013

The Epistemological Thought of Otto Hölder

Hölder, Mach, and the Law of the Lever: A Case of Well-founded Non-controversy

Oliver Schlaudt



Electronic version

URL: <https://journals.openedition.org/philosophiascientiae/830>

DOI: 10.4000/philosophiascientiae.830

ISSN: 1775-4283

Publisher

Éditions Kimé

Printed version

Date of publication: 1 March 2013

Number of pages: 93-116

ISBN: 978-2-84174-620-0

ISSN: 1281-2463

Electronic reference

Oliver Schlaudt, "Hölder, Mach, and the Law of the Lever: A Case of Well-founded Non-controversy", *Philosophia Scientiæ* [Online], 17-1 | 2013, Online since 01 mars 2016, connection on 28 juin 2022. URL: <http://journals.openedition.org/philosophiascientiae/830> ; DOI: <https://doi.org/10.4000/philosophiascientiae.830>

Tous droits réservés

Hölder, Mach, and the Law of the Lever: A Case of Well-founded Non-controversy

Oliver Schlaudt

Universität Heidelberg (Germany)

Laboratoire d'Histoire des Sciences et de Philosophie,
Archives H. Poincaré, UMR 7117, Université de Lorraine –
CNRS, Nancy (France)

Résumé : Nous analysons la réponse qu'a donnée Otto Hölder à la célèbre critique de la preuve archimédienne de la loi du levier par Ernst Mach et nous la situons dans sa pensée épistémologique. Pour Hölder, la preuve de la loi du levier fournit une étude de cas sur la déduction en mécanique. Considérant la preuve comme un processus constructif et synthétique montant d'un niveau conceptuel à un niveau supérieur, Hölder parvient à défendre la preuve d'Archimède contre la critique de Mach, pour qui au contraire une preuve n'amène qu'à une vérité analytique.

Abstract: Otto Hölder's reply to Mach's renowned critique of the Archimedean proof of the Law of the Lever is analyzed and contextualized as part of Hölder's epistemological theory. The proof of the Law of the Lever provided for Hölder a case study on deduction in mechanics. He developed in it an original understanding of proof as a constructive, synthetical process leading to higher order concepts. This enabled him to avoid the view held by Mach that proof can only lead to analytic truths, and to defend Archimedes against Mach's critique.

1 Introduction

The Law of the Lever was among the first laws of nature to be formulated in quantitative terms. It dates back at least to Archimedes' *On the Equilibrium of Planes* and possibly even to Aristotle's *Mechanics*. Shortly after its first formulation, scholars like Archimedes and Euclid were already seeking to prove it by means of deduction from general axioms and postulates. The Law of the Lever thus composes the very core of rational mechanics. However, in 1883 Ernst Mach accused Archimedes' proof of circularity, thereby provoking a discussion about the role of proof in mechanics, and indeed about the nature of

the foundations of mechanics as a whole. This discussion is associated first and foremost with the name of Giovanni Vailati, though Otto Hölder was, in fact, the first to reply to Mach's criticism.¹ Hölder was interested in the proof of the Law of the Lever as a case study on deduction in mathematical physics. He agreed with Mach on experience being the only source of knowledge. But unlike Mach, Hölder did not view deduction as a mere test of agreement between the premises and the conclusion: For Hölder, deduction constructively bridges a qualitative difference between the premises and the conclusion. He attempted to explicate this concept of deduction in terms of "synthesis".

In this paper, an outline of the discussion between Mach and Hölder will be given, with particular stress on Hölder's contributions and his epistemological thought. The paper consists of two parts: In the first part (Section 2), I shall present a rigorous analysis of Mach and Hölder's assumptions, aims, and strategies, given that they were at cross-purposes in this discussion, and they themselves never clearly elaborated on their conflict of opinion. It will be shown that Hölder tried to refocus the discussion on the role of proof in the mathematical sciences. In the second part (Section 3), some key aspects of Hölder's account of deduction in the mathematical sciences will be discussed. I will argue that Hölder provides a pertinent defence of Archimedes' proof against Mach's critique.

2 The Hölder-Mach-controversy

2.1 The Law of the Lever

In modern terms, the Law of the Lever is a special case of conservation of angular momentum: "If the total external torque is zero, then the total vector angular momentum of the system is a constant" [Feynman, Leighton & Sands 1963, 20–25]. When applied to the "lever", that is, to a rigid rod pivoted on a fulcrum, this is to state that the lever is in equilibrium if the weights put on it are reciprocally proportional to their distances to the fulcrum. This is a descriptive law which can, however, be translated into a normative rule by saying that reciprocal displacements of weights on both sides of the fulcrum do not disturb equilibria. In this paper, the normative version will often be more appropriate, since equilibrium-conserving manipulations of configurations of masses will be in the main focus.

The Law of the Lever can be sited for the first time in Aristotle's *Mechanics*, be it as an original element of the text or as a later addition. This text illustrates, as Beisenherz and Renn *et al.* pointed out, the recent appearance of the lever with unequal arms in everyday life in ancient Greece. This instrument allowed for determining weight by displacing a given counterweight on

1. Vailati already in 1897 published two papers on the proof of the Law of the Lever and its critics [Vailati 1897a], [Vailati 1897b]. These texts however do not contain any reference to Mach.

the opposite arm of the lever ([Beisenherz 1980, 455–456]; [Renn, Damerow & McLaughlin 2003, 47]). According to Renn *et al.*, the lever with unequal arms led to the generalization of the concept of equilibrium. This generalized idea accounted for the fact that weights evidently can be compensated not only by weights, but also by distances. Concept formations like “force of the weight” (Euclid) or “momentum ponderis” (Vitruius) reflected this idea [Renn, Damerow & McLaughlin 2003, 55]. I would like to add Stevin’s concept of “*evenstaltwichtigheid*” as opposed to “*evenwichtigheid*” (that is, in accordance with Dijksterhuis’s translation, “being of apparent equal weight” and “being of equal weight” respectively), introduced in his *De beghinselen der weeghconst* from 1586 [Stevin 1955, 38, 105]. Masses which compose a system in equilibrium are of equal apparent weight; equality of weight can be assigned to them only in the case that the system is a symmetric one. We can put this in modern terms by saying that the force F of a body is a function of the weight m and its position relative to the fulcrum d , $F = F(m, d)$. It is clear that F is a linear function of the weight m , since a (homogeneous) body m can be conceived as consisting of parts m_i whose forces together add to the total force $F(m) = \sum_i F(m_i)$. The way F depends on d is the critical question, because no similar reasoning holds for d : A mass m at distance d cannot be conceived as compounded of n parts of mass m at the distance $1/n \cdot d$. The Law of the Lever states that F is a linear function of d as well. However, it should be noted that all kinds of functions, such as a power law of the form $F = m \cdot d^n$, are compatible with the special case of the symmetric lever, i.e., the fact that equal weights can be displaced symmetrically without disturbing the equilibrium.

2.2 Archimedes’ proof of the Law of the Lever

Archimedes presented an early proof of the Law of the Lever, that is, situated it within a theory of deductive style. As Vailati [Vailati 1904], Dijksterhuis (1938, quotations from the second English edition [Dijksterhuis 1987]), and [Renn, Damerow & McLaughlin 2003] concordantly argue, the Archimedean proof was essentially based on the notion of the center of gravity, that is, according to Dijksterhuis, on the premise

that the influence exerted on the equilibrium by a body suspended on a lever is judged exclusively by the gravity of the body and the place of its center of gravity, and that the shape is immaterial [Dijksterhuis 1987, 291]

and consequently

that a lever equilibrium is not disturbed, if one of the suspended magnitudes is made to change its shape while preserving its weight and center of gravity. [Dijksterhuis 1987, 292]

As Dijksterhuis made clear, the above was intended to be a reconstruction of the actual cognitive content of the Archimedean theory [Dijksterhuis 1987,

295]. Hölder and Mach, however, were not interested in a historical reconstruction of Archimedean physics; they were attacking a systematic problem of rational mechanics. In direct terms, their question was whether the Archimedean proof was valid or not.² Accordingly, in the following I will disregard the question of in how far Hölder and Mach did justice to Archimedes. Rather I shall demonstrate the problem both attacked. In order to do so, I will first present Hölder's reconstruction of Archimedes' proof.

Consider a body of weight 10 fixed at position 0 (fig. 1-32). This body is successively cut into pieces of weight 1 which are symmetrically distributed

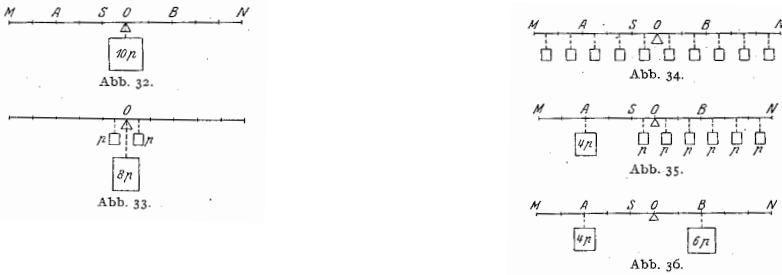


FIGURE 1: Hölder's reconstruction of the Archimedean proof. Figures 32-36 from [Hölder 1924, 40]

pairwise around position 0 in linearly increasing distances till a uniform distribution of ten unit weights is reached (fig. 1-34). In a next step, the four outer weights on the left hand and the remaining six weights on the right hand are symmetrically recombined at their respective geometric centers (fig. 1-35, 36). This finally produces two bodies of weights 4 and 6 at positions 3 and 2 respectively (fig. 1-36). The idea of the proof is as follows: Since the initial distribution (fig. 1-32) was in equilibrium, and since only symmetrical, i. e., equilibrium conserving manipulations, have been carried out, the final distribution (fig. 1-36) must still be in equilibrium. The final distribution however is no longer symmetrical: The distances are inversely proportional to the weights, a result which can easily be generalized for any pair of commensurable weights. It should be emphasized that the Archimedean proof does not deal with actual manipulations, but rather intends to rationalize the Law of the Lever in

2. As regards this question, it was disputed by [Beisenherz 1980] whether Dijksterhuis' reconstruction is a plausible candidate for a systematic defense of Archimedes. Beisenherz' main point is that Archimedean statics should be read as an attempt to establish a metrization of weight and that the center of gravity cannot be used within this attempt, since this notion depends on the metrization of weight [Beisenherz 1980, 457-459].

terms of manipulations “performed in thought”, i. e., manipulations performed by means of a graphical representation. Of course, this representation itself tacitly rests on the notion of the center of gravity, for it seems that in reality, the original weight of 10 at the beginning of the proof was not intended to be a physical body, but rather represents the center of gravity which remains unchanged in the following steps.

Before going into the details of the Mach-Hölder controversy, two remarks should be made in order to facilitate the discussion. These remarks concern features of the proof which will not be of importance to our discussion and thus can be dealt with in a short comment. First, it should be noted that the proof only applies to commensurable ratios of lengths and weights. Archimedes, however, generalized it to apply to both, commensurable and incommensurable ratios. The generalized proof essentially consists of a reduction of the problem of incommensurable ratios to commensurable ratios by subtracting an appropriate amount of one of the weights, thus completely resting on the simpler case. Since the point of the whole discussion regards a step of the basic proof for the case of commensurable quantities, I will not refer to the extension to the case of incommensurable quantities in the following. Second, there are three further suppositions tacitly made in the proof which should be mentioned: (1) that the weights are made of a homogeneous material, (2) that the lever is perfectly rigid and of mass zero, and (3) finally that the lever is put into a homogeneous gravity field. All three suppositions may be classified as idealizations not uncommon to mechanics. As regards the third point, concerning the gravity field, Hölder himself parted with the supposition of homogeneity by abstracting from the origin of the acting forces. In fact, he mostly did not speak about weight, but simply about acting forces in general.³ These suppositions are also not of interest to the purpose of the present paper and will not be mentioned again.

2.3 Mach’s critique of the proof

Archimedes, as Hölder and Mach understood him, claimed to establish the Law of the Lever by means of a special case, namely by showing that an asymmetric configuration can be reached based on a symmetric one simply by distributing and recombining weights symmetrically. In his *Die Mechanik*

3. Palmieri recently took up this issue in a paper from 2008, showing that the actual form of the Law of the Lever is contingent on the specific form of the earth’s gravity field. I believe Hölder and Mach were perfectly aware of this. In his response to Palmieri, Van Dyck additionally provides evidence that the problem of the uniqueness of the center of gravity of bodies on the curved surface of the earth was already discussed in the sixteenth century by Simon Stevin and Guidobaldo del Monte [Van Dyck 2009, 318]. Palmieri’s overall aim in his two papers, [Palmieri 2008], [Palmieri 2009] is to defend Mach’s rejection of non-empirical sources of knowledge. In light of the Mach-Hölder debate, however, it becomes clear that this point actually was not at issue.

in ihrer Entwicklung, historisch-kritisch dargestellt from 1883, Ernst Mach claimed that such a proof cannot be sound. In fact, the theorem to be proven, i. e., the Law of the Lever, must have been tacitly presupposed—otherwise it would have been impossible to derive it from the given premises, dealing only with the special case of symmetric distributions. Mach analyzed the proof and accurately localized the point where tacit presupposition enters into the proof. According to Mach’s analysis, the first half of the proof (leading from fig. 1-32 to 1-34) is sound. Indeed, the weights are displaced symmetrically with regard to the fulcrum. In the second step (fig. 1-35, 36), on the other hand, the weights are recombined symmetrically with regard to a point located beside the fulcrum. Such an operation is feasible only if an additional condition is fulfilled (fig. 2). The two weights together at point d (the distance from

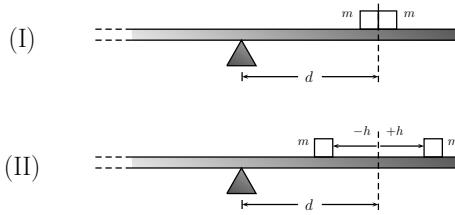


FIGURE 2: Mach’s analysis of the Archimedean proof.

the fulcrum) must exert the same force on the lever as both weights when separately located at the points $d-h$ and $d+h$ (that is, the effects of displacing them from position d to $d-h$ and $d+h$, respectively must compensate for, or cancel out, each other). In mathematical terms, the following functional equation must hold:

$$2m \cdot f(d) = m \cdot f(d-h) + m \cdot f(d+h). \quad (6)$$

It is then easy to show that this implies that f is a linear function of d , i. e., that $f(d) = \alpha \cdot d$, and hence that the presupposition is equivalent to the Law of the Lever.⁴ Thus, Mach concludes, the proof is circular.

4. I should mention that the first to use these mathematical terms was actually Vailati in his article from 1904. However, from Mach’s texts it is clear that the argument presented here was the one intended by him and that he was aware of the fact that the compensation of the effects of a symmetric displacement beside the fulcrum is equivalent to the Law of the Lever—otherwise his arguments would make no sense. I stress this point because of a paper published by Palmieri in 2008 containing circumstantial proof of the equivalence in question, where the author claims to be the first one to make this mathematical argument. Actually, Hölder explicitly indicated how to prove the equivalence in *Die mathematische Methode* [Hölder 1924, 44–45, note 1].

2.4 Hölder's defense of Archimedes' proof

In one of his extensive annotations to *Anschauung und Denken in der Geometrie*, Hölder's inaugural lecture at Leipzig held on June 22, 1899, Hölder responded to Mach's critique. In that lecture, he explicitly denied that the Archimedean proof presupposes the law to be proven:

Mach advances the view that in the Archimedean Proof it is already presupposed that the effect that a weight exerts on the lever depends on the product of the magnitudes of the weight and the arm of the lever. This is not the case. [Hölder 1900, 63, Eng. trans. 44]

In order to defend the Archimedean proof, Hölder came up with a detailed analysis of its premises, one which is, I believe, much more subtle than what one finds in Mach and thus deserving a closer look. Hölder, in accordance with Mach, identified the symmetric replacement beside the fulcrum as the problematic point. According to Hölder, this substitution can be "accounted for" based on two arguments: first of all, through experiences with symmetric scales; second by the "assumption", as Hölder puts it, "that the observed equivalence [i. e., on the symmetric scale] also holds true under different circumstances". "Under different circumstances" here clearly means: beside the fulcrum. Hölder continues in his analysis and finds that the assumption that symmetric displacements are also feasible beside the fulcrum can be reduced to the following:

a system of forces acting on a rigid body is in equilibrium, if a part of the forces considered separately is in equilibrium and if the same also holds of the other part of the forces. This is a postulate (an axiomatic assumption)—motivated, of course, by experience. [Hölder 1900, 64, Eng. trans. 44]

Hölder thus reduces the second tacit assumption of the Archimedean proof to the superposability of equilibria, as he explicitly makes clear elsewhere [Hölder 1931, 322]. To summarize, Hölder seeks to rationally reconstruct the Archimedean proof based on the following two assumptions which serve as its premises:

- (P1) Symmetrical configurations are in equilibrium and displacements symmetric to the fulcrum do not disturb equilibria. That is, in fig. 3: Displacement $(I^*) \leftrightarrow (II^*)$ is equilibrium conserving.



FIGURE 3: Premise 1 in Hölder's reconstruction.

(P2) Equilibria can be superposed. That is, in fig. 4: If configuration (I**) is in equilibrium, configuration (II**) also is in equilibrium, and vice versa.

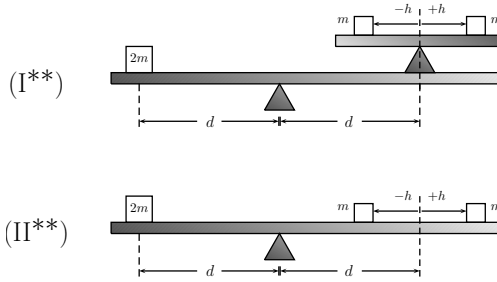


FIGURE 4: Premise 2 in Hölder's reconstruction.

It is clear that these two premises lead to a generalization of the case of the symmetric lever. According to the premises, symmetric displacements of equal weights are feasible at *any* point on the configuration, and not only when symmetric with regard to the fulcrum. The second premise is equivalent to the assumption that any subsystem can be reduced to its center of gravity. In Hölder's formulation, this premise is particularly striking, because, when read backwards, it yields a kind of geometrical or even graphical algorithm for the analysis of matter distributions. Applying these rules (in the order P2-P1-P2) yields a justification for the critical step in the proof of the Law of the Lever by showing the configurations of fig. 5 to be equivalent pairwise.

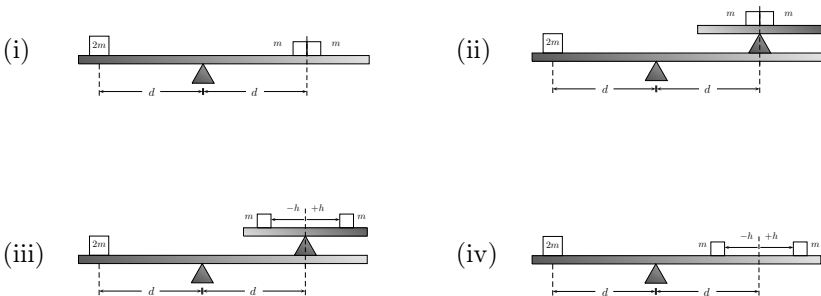


FIGURE 5: Hölder's analysis of the step (I) \rightarrow (II) in fig. 2.

It seems that Hölder was elaborating here on what Archimedes actually did in his proof, i. e., picking out subsystems and manipulating them symmet-

rically with regard to their respective center of gravity. In this way, Hölder accounted for the equivalence of (I) and (II) in Mach's proof (fig. 2) through the equivalence of (I**) and (II**) in (P2), i. e., the assumption that any subsystem can be reduced to its center of gravity. However, as Mach made clear, this premise is precisely equivalent to the Law of the Lever, as becomes obvious when spelled out in mathematical terms as done above in the functional equation (6). The proposition P3 (comprising P1 and P2)

(P3) Displacements of *equal* weights are feasible (i. e., equilibrium conserving) when symmetric to *any* point on the configuration.

is strictly equivalent, as is established in the proof of the Law of the Lever, to the proposition

(P4) Displacement of weights is feasible (i. e., equilibrium conserving) when, with regard to the fulcrum, reciprocal to the weights.

The latter proposition is, of course, the Law of the Lever. Hölder's analysis of the premises is certainly more subtle than Mach's, but in how far then does it challenge Mach's criticism? Here is the conclusion Hölder himself drew from his analysis:

Whatever one might think about Archimedes' suppositions [P3], it cannot be denied that it is not possible to immediately, i. e., by means of the mere power of judgment, perceive in them the Law of the Lever [P4], though this metrical law necessarily follows from these suppositions in a mathematical reasoning. [Hölder 1900, 65, Eng. trans. 44]

Hölder hence willingly admits that the Law of the Lever follows from the premises. This indeed amounts to nothing other than stating that the proof is sound. Hölder's point is clearly a different one. He does not deny that the conclusion is contained in the premises—and how could he have done this? On the contrary, Hölder emphasizes the way the conclusion is contained in the premises: The law to be proven is contained in the premises in such a way that it is impossible to conceive it through “the mere power of judgment”. It is difficult to say what this power of judgment consists in exactly, but if one accepts talk of “containment” of the conclusion in the (conjunction of the) premises—as did Mach, Hölder, and Vailati—one arrives at a peculiar diagnosis for the Mach-Hölder controversy: Clearly, both, Hölder and Mach, admitted that the conclusion, i. e., the Law of the Lever, is implicitly contained in the premises, but each drew contrary consequences from this. Mach states that the conclusion is (implicitly) contained in the premises—hence the proof is circular. Hölder, on the other hand, says that the conclusion is (implicitly) contained in the premises—hence the proof is sound.

2.5 A failed dialogue

The paradoxical diagnosis characterized above is most certainly due to diverging conceptions of logic—Mach still thought of syllogistic logic, whereas Hölder

already approached mathematical logic—and thus to the misleading metaphor of “containment”, a defect which we will later see how to avoid in the reconstruction of Hölder’s argument (Section 3.2). But there is a different point I would like to draw attention to: besides diverging conceptions of logic, at least a partial explanation for the peculiar incommensurability of Hölder and Mach’s arguments can be found in the fact that both had completely different questions in mind when tackling the problem of deduction in mathematical physics, illustrated by the Archimedean proof of the Law of the Lever. When discussing this proof, both made clear what their intentions respectively consisted of. For Mach, the leading question of his critical survey of mechanics was whether there are non-empirical sources of knowledge. The answer he gave was an emphatic “No!”, as he repeatedly highlights:

But the aim of my whole book is to convince the reader that we cannot make up properties of nature with the help of self-evident suppositions, but that these suppositions must be taken from experience. [Mach 1915, 2, 7th German edition]

If we were unable philosophically and *a priori* to excogitate the simple fact of the dependence of equilibrium on weight and distance, but were obliged to go for *that* result to experience, in how much less a degree shall we be able, by speculative methods, to discover the *form* of this dependence, the proportionality! [Mach 1919, 14, 4th German ed.]

Mach’s point clearly is that experience is the only source of knowledge and that linking propositions by means of deduction cannot change that fact. Hölder, however, completely agreed with Mach on the empirical nature of scientific knowledge, a point which he repeatedly emphasized: he more or less took it for granted that all scientific knowledge is empirical;⁵ Hölder instead raised the question of what, then, is the purpose of deduction and deductively organizing empirical knowledge. In his comprehensive study *Die mathematische Methode* from 1924, Hölder programmatically declared:

To clarify deductive reasoning by means of analyzing examples is the main concern of my treatise. [Hölder 1924, 3]

And already in 1900 he had stressed that this aim can be achieved without touching epistemological questions about the source of our knowledge, i.e., the kind of questions Mach was interested in [Hölder 1900, 8, Eng. trans. 19]. This discrepancy in Hölder’s and Mach’s aims might be at least a partial explanation of why they could end up with such contrary conclusions, notwithstanding a quite considerable agreement on the premises, namely that the Law of the Lever is implicitly contained in Archimede’s assumptions, and that these assumptions as well as the law deduced from them, are of empirical character.

5. In *Anschauung und Denken* Hölder takes both intuition and experience into consideration as sources of knowledge, especially for geometric axioms [Hölder 1900, 57, Eng. trans. 41], and finally advocates for the latter [Hölder 1900, 72, Eng. trans. 47].

Hölder tried to refocus the discussion toward the more important and more promising issue of the function of deductively ordering empirical knowledge. With regard to Mach, however, Hölder failed: Mach, though he appreciated Hölder's critique⁶ and to some extent adopted Hölder's terminology,⁷ did not readjust the focus of his critique, and his response to Hölder in the fourth edition of *Die Mechanik* did not provide any substantial argument:

I cannot agree with O. Hölder who upholds the correctness of the Archimedean deductions against my criticism in his essay *Anschaung und Denken in der Geometrie*, although I am greatly pleased with the extent of our agreement as to the nature of the exact sciences and their foundations. It would seem as if Archimedes (*De aequiponderantibus*) regarded it as a general experience that two equal weights may under all circumstances be replaced by one equal to their combined weight at the center (Theorem 5, Corollary 2). In such an event, his long deduction (Theorem 6) would be [un]necessary [my correction of the translation, O. S.], for the reason sought follows immediately (see pp. 14, 513). Archimedes' mode of expression is not in favor of this view. Nevertheless, a theorem of this kind cannot be regarded as *a priori* evident; and the view advanced on pp. 14, 513 appears to me to still be uncontroverted. [Mach 1919, 514–515]

Here Mach, when referring to Archimedes' "mode of expression", provided more philological than systematical evidence and did not at all challenge Hölder's point. In particular, Mach did not attempt to analyze the origin of his dissent from Hölder. The controversy thus ended before having revealed its real subject matter.

2.6 The real issue: Mach and Hölder on logic and proof in mathematical physics

What, then, should have been the substance of the Mach-Hölder-controversy? It is clear from the preceding analysis that the underlying argument between

6. In the preface to the fourth edition, Mach underlined the following: "New books and criticisms touching on my discussions have received attention in special additions, which in some instances have assumed considerable proportions. On these strictures, O. Hölder's note on my criticism of the Archimedean deduction (*Denken und Anschauung in der Geometrie*, p. 63, note 62) has been of special value, inasmuch as it afforded me the opportunity of establishing my view on still firmer foundations (see pages 512–517). I do not at all dispute that rigorous demonstrations are as possible in mechanics as in mathematics. But with respect to the Archimedean and certain other deductions, I am still of the opinion that my point is the correct one" [Mach 1901, xi]; [Mach 1919, xv–xvi].

7. This can be seen in the use of the phrases "the actions of two equal weights to be the same *under all circumstances* as that of the combined weights" [Mach 1919, 513] (my italics) and "general experience" [Mach 1919, 514].

Mach and Hölder is not about non-empirical sources of knowledge, but about the interest in structuring empirical knowledge in a deductive manner. Mach and Hölder had different ideas about the role of logic and deduction, indeed.

Mach was quite clear about what he conceives as the role of logic. He developed his account of deduction in a short article, “Deduktion und Induktion in psychologischer Beleuchtung”, published as a chapter of *Erkenntnis und Irrtum* in 1905. It is there that he essentially repeated Mill’s critique of syllogistic reasoning, i. e., that the general premise cannot be held without already asserting the conclusion. He stressed that syllogisms hence do not provide new knowledge, though one might have that impression: only experience is able to provide knowledge. He comes to the following conclusion:

This alone shows that the opening up of new sources of knowledge cannot be the task of the rules of logic, which rather serve only to examine whether findings drawn from other sources agree or disagree, and, if the latter, to point to the need to secure full agreement. [Mach 1976, 225]

Here Mach offers an instrumentalist account of logic, the latter serving as a mere corrective: In particular, there are no proofs in physics for Mach. Logic does not uncover truth; it only indicates disagreement and thus error when applied to knowledge drawn from experience.

Hölder, as we have seen, agrees with the point that experience is the only source of knowledge. He nevertheless remains at odds with Mach as regards the role of proof and deduction. His arguments are not self-evidently drawn from the discussion with Mach since they failed to push the analysis of their dissent that far. It at least seems to be clear, though, that Hölder, when claiming that it is not possible, through the mere power of judgment, to immediately perceive the Law of the Lever in the premises of the Archimedean proof [Hölder 1900, 65, Eng. trans. 44], directly answers Mach’s argument that a syllogism’s conclusion “could have directly been seen” in the premises [Mach 1976, 227]. There is indeed a major difference between the proof of the Law of the Lever and the ordinary syllogisms Mach refers to. Certainly it is impossible to maintain that all men are mortal without already asserting that (the man) Socrates is mortal. It is also true that the conclusion of this syllogism can directly be seen in the general premise, the latter being understood simply as a conjunction of all special cases. But does the same hold true for the proof of the Law of the Lever? Hölder, being one generation younger than Mach, indeed referred to Jevons’ “Logic of relatives” which he thought more adequate than Aristotelian syllogisms to capture mathematical inferences in general [Hölder 1900, 41, n. 25, Eng. trans. 34]. As regards the Law of the Lever, too, Hölder clearly believes that its proof cannot be modeled on a syllogism. Deriving the Law of the Lever from its premises as Hölder reconstructed them (P1 and P2) takes work. Vailati, who also used the metaphor of containment, made this clear in the following impressive illustration:

Finally, the assertion that in Archimedes’ premises there is already implicitly contained the conclusion that he deduces from

them, seems to me true only in the sense in which it is true of any mathematical proof, insofar as in any mathematical proof the truth of the proposition being proved appears as a simple consequence of certain operations of selection, linking, and coordination performed on fundamental propositions that are the basis of the treatment. And this, to use a celebrated Aristotelian paragon, is as far from diminishing the value of their discovery and of their proof as it would be far from diminishing the value of a sculptor to claim that the statue he created was already within the marble from which he crafted it, and that he has done nothing but to remove the superfluous parts from the block of marble.
[Vailati 1904], quotation from the English translation in [Palmieri 2009, 321]

We see, then, that Vailati agrees with what I have worked out to be Hölder's main point,⁸ as did Dijksterhuis, it should be mentioned.⁹ But in order to emphasize the strength of Hölder's perspective, I should restate the issue in more abstract terms.

We saw that Mach thought logic to merely provide a check for consistency. For example, if two empirical thoughts are in logical contradiction, at least one of them must be false. However, "agreement" and "disagreement" of empirical knowledge are symmetric relations, whereas deduction establishes a hierarchy which is essentially asymmetric. Hölder took this hierarchy seriously and could therefore use it in defending the Archimedean proof. This hierarchy indicates a qualitative difference between the premises and the conclusions—a difference which is bridged in deduction. This is precisely what Hölder was after. Put into Vailatian terms, Hölder's main concern was to explicate what it means to get the statue out of the marble block, that is, to explicate (1.) in what the difference between the premises and the conclusion consists and (2.) how this difference is bridged in deduction. I will study Hölder's answers to these questions in the next section.

3 Hölder's account of deduction in physics

The hierarchy between the premises and the conclusion must result from a difference between them. In general, it is not easy to say what this difference actually consists of, though a first guess would be generality. But given

8. In fact Vailati already expressed this point in abstracto in his 1898 paper on deduction [Vailati 1898, 690].

9. Dijksterhuis also shows that the Law of the Lever is the solution to equation (6) and remarks: "Indeed, if it is not necessary to solve the functional equation in this case, when is there any sense in doing so? And if in general the drawing of a conclusion Q from a group of premises P may be called superfluous when P applies only if Q is true, one may just as well reject as superfluous all mathematical proof, for if Q is not true, it is clear that P will not hold!" [Dijksterhuis 1987, 294].

Poincaré's discussion on the principle of recurrence [Poincaré 1902], the answer that the premises are simply more general must be ruled out as naive. This was held true by Poincaré at least for arithmetical proofs involving induction. But in the case of the proof of the Law of the Lever, it may also hold that the premises (P3 in my reconstruction given above, p. 101) are not necessarily more general than the conclusion (P4).¹⁰ Hölder's answer is that the difference lies in degrees of epistemic simplicity. This will be explained in Section 3.1. In deduction, this difference is then bridged by synthesis, which Hölder parallels to, or even identifies with, deduction, as will be explained in Section 3.2. It will then be shown in Section 3.3, that this account holds true, according to Hölder, not for the whole of physics, but only for those parts which can be reconstructed on the basis of axioms expressing simple experiences. Aside from these parts, there are different domains which do not rest on axioms, but on hypotheses.

3.1 Axioms in mechanics

As I have already emphasized, Hölder, like Mach, believed both premises and conclusions, to be empirical. Hölder's analysis of deduction thus had nothing to do with the quest for non-empirical sources of knowledge, the main target of Mach's empiricist criticism. However, as already seen in *Anschauung und Denken in der Geometrie* from 1900, Hölder suggested that the premises and the conclusion might be the result of different kinds of experience, which will then turn out to be his main point about the difference between axioms and theorems:

already the experience of every-day life might lead us to postulate the Archimedean assumptions [P1 and P2], which recommends these assumptions as the simpler foundations. [Hölder 1900, 65, Eng. trans. 44]

The difference between the premises and the conclusion is hence reduced to the difference between two different kinds of experience that they originate from; the former are the result of everyday experiences, whereas the latter require a somehow more demanding kind of experience. Hölder took up this issue in several of his writings [Hölder 1914, 1924, 1931] from which it is possible to achieve a quite precise picture of what he had in mind when distinguishing between different kinds of experience.

In *Die Arithmetik in strenger Begründung* of 1914 Hölder gives an important hint when using the tacit premises of the Law of the Lever worked out in *Anschauung und Denken* in order to provide a metrization of weight—an issue also present in *Anschauung und Denken* when discussing the problem of the rigid body, a prominent question in the contemporary discussion on the

10. P3 holds true for any given point, but is restricted to equal weights; P4, in turn, holds for any given pair of weights but is restricted to the fulcrum as the point of reference.

foundations of geometry and measurement.¹¹ It appears that Hölder did not work out the details of this idea;¹² but from his remarks it already becomes clear, that the difference between quality and quantity is somehow supposed to be in line with the difference between the premises in question and the conclusion, the Law of the Lever, for metrization consists in establishing quantity on a fundament which is not quantitative. In *Die mathematische Methode* from 1924 Hölder finally explicitly applies the terms of quality and quantity in order to make clear his point and to definitively refute Mach's critique:

It should also be mentioned that the assumptions were not of quantitative, but of purely qualitative nature [...]. Hence the criticisms Mach directed against the famous Archimedean proof of the Law of the Lever should be considered as invalidated. [Hölder 1924, 318]

It is thus clear that Hölder considered deduction of the Law of the Lever to be the same as establishing a metrical law on a purely qualitative fundament. This fundament consists in the qualitative axioms first that symmetrical configurations are in equilibrium and second, that equilibria can be superposed, while the conclusion states in quantitative terms that, in equilibrium, length is inversely proportional to weight. In order to discuss this more closely I quote the premises as Hölder explained them in 1931, when touching on the problem for the last time [Hölder 1931, 322]:

1. Two forces of equal direction and equal magnitude, when acting on a rigid body, can be equilibrated by a force of opposite direction and twofold magnitude acting on the body at the midpoint between these two forces.
2. Equilibria, when superposed, result in equilibria.
3. Two forces acting on a body at the same point are equivalent to their sum when they are coming from the same direction, and cancel when they are coming from opposite directions.

It is obvious that these axioms do not dispose of quantity *tout court*. They presuppose quantitative notions of force (or, in the more special case of Archimedean statics, of weight) and of distance. Taken at face value Hölder's claim is hence wrong. But apart from the implausibility of such an obvious mistake there are other reasons for considering Hölder's claim more carefully. For there is indeed a difference between the premises and the conclusion which is linked to establishing a metrical law: The premises take length and weight (force) into consideration separately while the conclusion contains a statement about their functional dependence. Thus the issue is not establishing single quantities, as indicated in the 1914 paper [Hölder 1914], but establishing functional dependencies between different quantities. The confusion

11. [Hölder 1900, 5, 30, Eng. trans. 17, 30]; [Hölder 1924, 370–371], where he refers to [Zindler 1889]; [Dingler 1925, 310–330], and, according to [Heath 1926, vol. III, 226–227], [Veronese 1891]; etc.

12. Unlike [Beisenherz 1980], who is quite in agreement with Hölder's approach.

between establishing quantity and establishing quantitative relations becomes less problematic when one takes into account that Hölder, as we will see, regarded them as successive steps in the establishment of mechanics. These steps fit into Hölder's larger idea of the make-up of mathematical sciences which he circumscribed by the notion of synthesis, as will be explained next.

3.2 Deduction and synthesis

In his comprehensive study *Die mathematische Methode* from 1924 Hölder developed an overall view of mathematical sciences with which his considerations on the Law of the Lever fit well. The Law of the Lever undeniably plays the role of a case study for Hölder's approach. The correlation with geometry, for example, is striking in the following quotation:

Geometry also develops *metrical* dependencies based on certain simple facts concerning equality and some other relations. [Hölder 1924, 45]

In this book Hölder develops a general theory of concept formation by means of "synthesis". He adopted this term from his former teacher Paul du Bois-Reymond who used it synonymously with "*Construction*" (construction) and "*Aufbau*" (built-up) [Hölder 1900, 71–72, Eng. trans. 47], [du Bois-Reymond 1890, 14]. In [Hölder 1924, 5–6], synthesis consists of the construction of new concepts with the goal to identify and to substantiate hidden properties of already known concepts. Synthesis is clearly paralleled to, or even identified with, deduction which, as we have seen, also brings out implicit knowledge:

The construction [*Aufbau*] of concepts here even constitutes a part of the proof. [Hölder 1924, 5]

In geometry and mechanics, as Hölder explains, the concepts synthesized are constructed on the basis of intuitive or empirical concepts (as "point", "straight line", "plane", "angle", "force", "mass"). These concepts serve as a basis for a conceptual "super-structure" achieved by way of synthesis. Hölder stresses that this super-structure can again serve as a basis for iterated acts of synthesis ([Hölder 1924, 318]; cf. also the hints given in [Waerden 1938, 100]).

Hölder then expands this view to all mathematical disciplines, including arithmetic, geometry, and mechanics, by distinguishing between pure synthetic and hypothetico-synthetic concepts, [Hölder 1924, 295]. The former are products of an intellectual activity such as setting and canceling, unifying and separating, sequencing and coordinating [Hölder 1924, 296]. In particular, they yield numbers and operations on numbers [Hölder 1924, 161ff.]. With hypothetico-synthetic concepts, on the other hand, something empirical adheres to them insofar as the elements they are constructed from have an empirical meaning and insofar as the axioms involved are "occasioned", as Hölder states, by experience [Hölder 1924, 295]. The counterparts to intellectual activities are geometrical and physical operations such as the concatenation of quantities [Hölder 1924, 297]. Hypothetico-synthetic concepts in particular

are the concept of measure in general [Hölder 1924, 51], the measure of the arm of the lever [Hölder 1924, 318], and the concept of the center of gravity [Hölder 1924, 294]. The simple elements which are used in the construction or synthesis of the concept of measure in particular are the relation of order, equality, and addition [Hölder 1924, 67]. As regards the elements of mechanics, Hölder makes clear that both logical deduction and a formalist treatment fail (on Hölder's critique of formalism in general cf. [Radu 2003]), thus it holds true that they have to be exemplified in the empirical context of their original abstraction [Hölder 1924, 403]. He stressed that this context is situated in daily life and paid special attention to manual production procedures like the so-called three-plate procedure in the production of steel planes.¹³

The distinction between intellectual and physical operations as the basis of arithmetic on the one hand and geometry as well as mechanics on the other hand makes sense. Indeed, in counting, one brings together the objects counted simply by successively considering them. For example when counting the planets of the solar system, the actual planets may stay where they are; it is sufficient to consider them as planets in order to count them and to determine their number. Counting thus involves purely intellectual activity and no causal manipulation of the objects counted. In measurement, on the other hand, where arithmetic is somehow put into practice, things are much more complicated, for measurement is something more than counting: It is counting so as to determine the magnitude of a property. Here the objects really (physically) have to be brought together in a specific way so as to concatenate, say, two unit quantities to a quantity of double magnitude. In order to do that, one basically needs an empirical operation of comparison and an empirical operation of addition.

To conclude, we recognize two synthetical steps in the formulation of the Law of the Lever, leading from qualitative to quantitative notions and from quantitative notions to metrical laws correlating two quantities:

basic layer	qualitative concepts (<i>Lebenswelt</i>)	order, equality, addition
first layer	quantitative concepts	weight, distance
second layer	metrical laws and concepts comprising several quantities	center of gravity, measure of the lever $f(m, d) = m \cdot d$

Weight and distance are two quantitative concepts synthesized from the re-

13. [Hölder 1924, 370]. This idea is known from [Helmholtz 1887], [Helmholtz 1977] and, in particular, from Hugo Dingler. Hölder however does not refer to Dingler, but to [Hjelmslev 1916]. This procedure allows for producing planes without the need for a prototype. For an English description, cf. Wolter's note to the English translation [Dingler 1988, 385].

spective physical operations of comparison and addition. Adding the simple, pre-metrical fact that equilibria can be superposed, yields, again by way of synthesis, the Law of the Lever, codifying knowledge that before was somehow hidden in the simple elements. In Archimedes, this knowledge enters into the concept of the center of gravity, and it is interesting to see that scientific concepts are not mere composites of arbitrary conceptual elements, but actually codify empirical knowledge. Hölder did not work with the concept of the center of gravity. In Hölder, the knowledge codified in this concept rather provided a kind of (representational or graphical) rule in the analysis of equilibria which is then used in the proof of the Law of the Lever. This rule, however, brings about the same result, since it allows for the reduction of arbitrary subsystems to their respective center of gravity. In establishing the Law of the Lever, two quantities—mass and length—are thus tied together by forming a new concept—center of gravity—which is constituted from both and which comprises their relation, i.e., the Law of the Lever.

It is not inappropriate to refer to this kind of concept formation as “synthesis”. Moreover, I believe that Hölder’s account, by which deduction is identified with synthesis, indeed challenges Mach’s critique of the proof of the Law of the Lever. We saw that both, Mach and Hölder, used the perhaps misleading metaphor of “containment” of the conclusion within the premises. Here is a definition of circularity from more recent literature which avoids this metaphor:

What is standardly taken to make an argument circular is, that every reason for doubting the conclusion is an equally strong reason for doubting one or more of the premises or inferences which lead up to the conclusion. [Detlefsen 1986, 60]

Applying this to Hölder’s account indeed shows that reasons for doubting a proposition involving synthesized concepts of level n (testing the Law of the Lever in measurement) are not equally strong reasons for doubting the premises which are built from synthesized concepts of level $n - 1$ and which thus rest on a simpler, less sophisticated kind of experience (the qualitative experience expressed in the premises P1 and P2). This seems to me to be the argument behind Hölder’s short commentary from *Die mathematische Methode* already quoted above (cf. p. 107), namely that the assumptions of the proof are of purely qualitative nature and that hence Mach’s critique is invalid [Hölder 1924, 318]. Therefore, even if Mach is right in that the proof of the Law of the Lever does not change our epistemological situation (in the sense that it does not dispose of experience, but merely shows different experiences to be compatible), Hölder succeeds in showing that our discursive situation definitively changes (for the proof provides stronger arguments by linking given experiences to experiences of a simpler kind).¹⁴

14. It seems that Van Dyck, in his defense of Vailati against a misreading by Palmieri [Palmieri 2008], has a quite similar point in mind when explaining that, according to Vailati, the Law of the Lever is “a more informative way to describe

3.3 Deduction and explanation

The above account of deduction in terms of synthesis, however, was not meant to hold true for the whole of physics. Indeed, there is evidence in *Die mathematische Methode* that Hölder admitted a special status to the Law of the Lever and mechanics in general within mathematical physics. This becomes evident when distinguishing more carefully between axioms and hypotheses or assumptions. In fact, it was common during Hölder's time to distinguish between hypotheses and axioms as regards their epistemological function and status. Nadine de Courtenay substantiated this distinction in a recent study in particular in Meinong, whose work was well known to, and casually cited by, Hölder [de Courtenay 2010]. De Courtenay reconstructs the distinction between hypotheses and axioms in terms of representational modes:¹⁵ "Assumptions and hypotheses both appear as indirect modes of presentation in contrast to the direct mode akin to axioms." The direct mode of representation arises, according to de Courtenay, from abstraction from the phenomena investigated while the indirect mode of representation depends on the means of description and may even lead to propositions contrary to the facts, in which case the corresponding representations cannot refer to any existing thing [de Courtenay 2010, 45]. De Courtenay stresses that the different modes of representation carry with them different conceptions of explanation: In contrast to axioms, hypotheses lead to more than a mere classification of facts, but "[imply] a causal relation: A stipulated external object or system, exemplifying the properties denoted by the alien system of concepts, is responsible for the causal production of the empirical phenomena observed" [de Courtenay 2010, 44–45]. The point regarding the modes of presentation fits quite well with Hölder's account, as will become clear. We have already seen that Hölder did not think of axioms as self-evident judgments which are neither empirically verifiable nor, in the case of mathematics, demonstrable. From his analysis of the Archimedean proof, it is clear that Hölder thought of axioms as expressions of simple empirical facts (whereby "simple" is to be understood relatively to the next layer of synthesized concepts). This does not only apply to Archimedean statics, but also hold true for an astonishingly large part of mechanics. Hölder explicitly mentions Newton's third law (equality of action and reaction) as well as d'Alembert's principle,¹⁶ which are considered to be abstracted from simple

what is allowed by [the premises of its proof]" and that "we can see Archimedes as using mathematics to reorganize the empirical information already at his disposal" [Van Dyck 2009, 316–317].

15. According to de Courtenay, there is a further distinction between hypotheses and assumptions which lacks a counterpart in Hölder. Hypotheses and assumptions, though they are on a par as regards the mode of representation, differ as to the mental attitude: "Assumptions and hypotheses encapsulate different attitudes of the subject towards his object: Assumptions do not involve belief, whereas hypotheses do involve belief" [de Courtenay 2010, 43].

16. Cf. also Jourdain's analysis [Jourdain 1912, 298–301] of Hölder's work on d'Alembert's principle and analytical mechanics [Hölder 1896].

experiences and then generalized by way of analogy [Hölder 1924, 430–431]. Hölder, however, recognizes a limitation in this method. In particular thermodynamics, electrodynamics, the theory of atoms, and relativity theory are said to rest on hypotheses which essentially differ from axioms [Hölder 1924, § 151, § 155–165]. According to Hölder, hypotheses

are assumptions which postulate a basically non-empirical relation, a relation often even in contradiction with experience to a certain degree. [Hölder 1924, 434]

Hypotheses postulate the existence of objects whose properties are commonly not found together in experience. So the atom, which has physical mass without being divisible, and the ether, which bears physical properties without mass. [Hölder 1924, 381, note 2]

Hölder's distinction between axioms expressing simple empirical facts and hypotheses postulating facts essentially beyond experience and intuition somehow fits with the distinction Vailati made between two ends of deduction, namely deduction as a means of proof and deduction as a means of research [Vailati 1898]. In the first case, deduction consists in reducing the disputable to the indisputable, that is, the axioms; in the second case, the indisputable phenomenon takes the place of the conclusion, which is linked by means of deduction to the hypotheses; hypotheses are essentially disputable, but serve to explain the phenomenon. Vailati thus probably conceives proof and explanation as an exclusive alternative, as did Hölder. Thus it is clear that the lessons drawn from the Law of the Lever were not intended to be generalized onto physics in general. Mechanics seems to hold a special status where deduction does not yield an explanation derived from non-empirical microscopic entities (postulated by hypotheses), but rather a synthetic construction of the theory derived from basic experiences (expressed in the axioms).

4 Conclusion

Hölder's account of deduction in mathematical physics is of interest for two reasons:

First he provided a more subtle analysis of the proof of the Law of the Lever than Mach. Mach had, indeed, called attention to a weak point of the proof; however, primarily due to the anti-apriorist motivation of his critique and his Aristotelian conception of logic, he ended up with the result that deduction in physics plays a rather marginal role since it does not create non-empirical knowledge, thus does not help disposing of experience. This result is plainly inadequate, for it does not explain why empirical knowledge is arranged in deductively structured theories at all.

The second point of interest is the theory of concept formation implied in Hölder's account of deduction. His theory of synthesis is admittedly difficult to understand, and perhaps not even consistently worked out in his magnum opus *Die mathematische Methode* from 1924; nonetheless, it offers a quite fascinating account. According to Hölder, concepts are synthesized either from empirical elements (including equality, order, etc.) or from formerly synthesized concepts. As such they have, as Hölder says, "hidden properties", brought out in further synthetical steps. I believe that this feature should be assessed in comparison with theories which explain concept formation in terms of nominal definitions. Hölder's analysis of deduction in mechanics provides a substantial argument for the view that concept formation is a much more complicated process than the amalgamation of an enumerable set of properties as one single notion for the sake of brevity. Hölder aimed at a reconstruction of scientific concepts on the basis of everyday linguistic and manual competences (as Hölder stresses in [Hölder 1900, 65, Eng. trans. 44] and [Hölder 1924, 403]), while paying special attention to manual production procedures like the so-called three-plate procedure in the production of steel planes [Hölder 1924, 370]. This account bears a certain resemblance to Hugo Dingler's approach (cf. [Dingler 1988]), notwithstanding the apparent incompatibility on a rhetoric level, namely empiricist in the case of Hölder and rationalist in the case of Dingler. Both argue for non-holistic semantics and hold quantitative concepts to be definable in (or synthesizable from) purely qualitative concepts. In particular, this involves the problem of defining the rigid body in qualitative terms, which Hölder explicitly considered possible without resorting to a solution in terms of conventions [Hölder 1924, 370–371]. Nevertheless there is also a strong affinity to Poincaré's conventionalist approach. For, unlike Dingler, Hölder stressed the role of experience in concept formation and agreed with Poincaré on the rejection of naïve empiricism: concept formations, according to Poincaré and Hölder, do not match or copy experience, but are "occasioned by experience". It thus becomes clear that Hölder pursued an original path between rationalism and empiricism fully deserving of attention.

Bibliography

BEISENHERZ, HEINZ GERHARD

1980 Archimedes und die Protophysik, *Philosophia Naturalis*, 18, 438–478.

DE COURTENAY, NADINE

2010 The epistemological virtues of assumptions: towards a coming of age of Boltzmann and Meinong's objections to the "the prejudice in favour of the actual"?, *Studies in History and Philosophy of Science*, 41, 41–57.

DETLEFSEN, MICHAEL

1986 *Hilbert's Program*, Dordrecht: Reidel.

DIJKSTERHUIS, EDUARD JAN

1987 *Archimedes*, Princeton: Princeton University Press.

DINGLER, HUGO

1925 Über den Zirkel in der empirischen Begründung der Geometrie, *Kant-Studien*, 30, 310–330.

1988 Method instead of epistemology and philosophy of science, *Science in Context*, 2(2), 369–408.

DU BOIS-REYMOND, PAUL

1890 *Über die Grundlagen der Erkenntnis in den exacten Wissenschaften, nach einer hinterlassenen Handschrift*, Tübingen: Laupp.

FEYNMAN, RICHARD P, LEIGHTON, ROBERT B. & SANDS, MATTHEW L.,

1963 *The Feynman Lectures on Physics*, vol. 1, Reading: Addison-Wessley.

HEATH, THOMAS L.

1926 *The Thirteen Books of Euclid's Elements*, vol. 3, Cambridge: Cambridge University Press, 2nd ed.

HELMHOLTZ, HERMANN VON

1887 Zählen und Messen, erkenntnistheoretisch betrachtet, in *Philosophische Aufsätze. Eduard Zeller gewidmet*, Leipzig: Fues's Verlag.

1977 Numbering and measuring from an epistemological viewpoint, in *Hermann von Helmholtz, Epistemological Writings*, edited by COHEN, R. S. & ELKANA, Y., Dordrecht: Reidel.

HJELMSLEV, JOHANNES

1916 Die Geometrie der Wirklichkeit, *Acta mathematica*, 40, 35–66.

HÖLDER, OTTO

1896 Über die Prinzipien von Hamilton und Maupertuis, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1896 (2), 122–157.

1900 *Anschauung und Denken in der Geometrie. Akademische Antrittsvorlesung gehalten am 22. Juni 1899 mit Zusätzen, Anmerkungen und einem Register*, Leipzig: Teubner, Eng. trans. *Intuition and Reasoning in Geometry* by P. Cantù and O. Schlaudt, this volume 15–52.

1914 *Die Arithmetik in strenger Begründung*, Leipzig: Teubner, 2nd edition, Berlin: Springer, 1929.

- 1924 *Die mathematische Methode. Logisch erkenntnistheoretische Untersuchung im Gebiete der Mathematik, Mechanik und Physik*, Berlin: Springer.
- 1931 Axiome, empirische Gesetze und mathematische Konstruktionen, *Scientia*, XLIX, 315–326.
- JOURDAIN, PHILIP E. B.
- 1912 The principle of least action. Remarks on some passages in Mach's mechanics, *The Monist*, 22(2), 285–304.
- MACH, ERNST
- 1883 *Die Mechanik in ihrer Entwicklung: historisch-kritisch dargestellt*, Leipzig: Brockhaus.
- 1901 *Die Mechanik in ihrer Entwicklung: historisch-kritisch dargestellt*, Leipzig: Brockhaus, 4th ed., 4., verb. u. verm. Aufl.
- 1915 *The Science of Mechanics. A critical and historical account of its development*, Chicago: Open Court, Supplement to the third English edition, containing the author's additions to the seventh German edition, transl. Philip E. B. Jourdain.
- 1919 *The Science of Mechanics. A critical and historical account of its development*, Chicago: Open Court, 4th ed., (transl. of the 4th German edition by Thomas J. McCormack).
- 1976 *Knowledge and Error. Sketches on the Psychology of Enquiry*, Dordrecht: Reidel.
- PALMIERI, PAOLO
- 2008 The empirical basis of equilibrium: Mach, Vailati, and the lever, *Studies in the History and Philosophy of Science*, 39, 42–53.
- 2009 Response to Maarten Van Dyck's commentary, including a translation of Vailati 1904, "the proof of the principle of the lever put forward by Archimedes in the first book of *Equilibrium of plane Figures*", *Studies in the History and Philosophy of Science*, 40, 319–321.
- POINCARÉ, HENRI
- 1902 *La Science et l'Hypothèse*, Paris: Flammarion, Eng. trans., *Science and Hypothesis*, London: Walter Scott Publishing, 1905.
- RADU, MIRCEA
- 2003 A debate about the axiomatization of arithmetic: Otto Hölder against Robert Graßmann, *Historia Mathematica*, 30, 341–377.
- RENN, JÜRGEN, DAMEROW, PETER & McLAUGHLIN, PETER
- 2003 Aristotle, Archimedes, Euclid, and the origin of mechanics: The perspective of historical epistemology, in *Symposium Archimedes*, edited by MONTESINOS SIRERA, J. L., Berlin: Max-Planck-Institut für Wissenschaftsgeschichte, *Preprint*, vol. 239, 43–59.

STEVIN, SIMON

- 1955 *Mechanics, The Principles Works of Simon Stevin. Edited by Ernst Crone, E. J. Dijksterhuis, R. J. Forbes, M. G. J. Minnaert, A. Pannekoek*, vol. I, Amsterdam: C. V. Swets & Zeitlinger, edited by E. J. Dijksterhuis.

VAILATI, GIOVANNI

- 1897a Del concetto di Centro di Gravità nella Statica di Archimede, *Atti della R. Accademia delle Scienze di Torino*, XXXII(13), 742–758, [reprinted in Vailati 1911, 79–90].
- 1897b Di una dimostrazione del Principio della Leva, attribuita ad Euclide, *Bollettino di Storia e Bibliografia Matematica*, nov.-dic., 21–22, [reprinted in Vailati 1911, 115–117].
- 1898 La méthode déductive comme instrument de recherche, *Revue de Métaphysique et de Morale*, IV, 667–703.
- 1904 La dimostrazione del principio della Leva data da Archimede nel libro primo sull'equilibrio delle figure piane, in *Atti del Congresso Internazionale di Scienze storiche, Roma 1903*, vol. XII, 243–249, English translation in [Palmieri 2009].
- 1911 *Scritti (1863-1909)*, Leipzig and Firenze: Johann Ambrosius Barth (Leipzig) and Successori B. Seeber (Firenze).

VAN DYCK, MAARTEN

- 2009 On the epistemological foundations of the law of the lever, *Studies in the History and Philosophy of Science*, 40, 315–318.

VERONESE, GIUSEPPE

- 1891 *Fondamenti di Geometria a più dimensioni e a più specie die unità rettilinee esposti in forma elementare. Lezioni per la Scuola di magistero in Matematica*, Padova: Tipografia del Seminario.

WAERDEN, BARTEL L. VAN DER

- 1938 Otto Hölder. Nachruf gesprochen in der öffentlichen Sitzung vom 25. Juni 1938, *Berichte der math.-phys. Klasse der sächs. Akademie der Wissenschaften*, XC, 97–101.

ZINDLER, KONRAD

- 1889 Beiträge zur Theorie der mathematischen Erkenntnis, *Sitzungsberichte der phil.-hist. Classe der K. Akademie der Wissenschaften zu Wien*, 118, 1–98.