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# Equality in the Presence of Apartness: An Application of Structural Proof Analysis to Intuitionistic Axiomatics

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**Abstract:** The theories of apartness, equality, and  $n$ -stable equality are presented through contraction- and cut-free sequent calculi. By methods of proof analysis, a purely proof-theoretic characterization of the equality fragment of apartness is obtained.

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# 1. Introduction

The idea of an *apartness* relation in place of an equality relation appears first in Brouwer's works on the intuitionistic continuum from the early 1920s. One of the basic insights of intuitionism was that the equality of two real numbers  $a, b$  is not decidable: The verification of  $a=b$  may require that  $a$  and  $b$  are computed to infinite precision. If  $a$  and  $b$  are positively distinct, instead, a finite computation is sufficient. Thus, Brouwer introduced the notion of apartness as a positive, intuitionistic basic concept. He also introduced special notations and terminology for the new intuitionistic concepts. We shall here use a uniform notation in which the intuitionistic notion is written with a slash over the classical one, as in  $a \neq b$ . The properties of this notion of apartness are, first, irreflexivity  $\neg a \neq a$ , and, secondly, the "splitting" of an apartness  $a \neq b$  into two cases  $a \neq c$  and  $b \neq c$ , for any middle term  $c$ . Intuitively, if  $a$  and  $b$  are positively apart, any  $c$  is apart from  $a$  or from  $b$ , for were this not the case with, say,  $a$ , it would positively be the case with  $b$ .

A notion of equality can be defined through negating apartness. Irreflexivity of apartness gives reflexivity of equality, and the "splitting" property has a contraposition that gives: If  $a=c$  and  $b=c$ , then  $a=b$ . This is the "Euclidean" formulation of transitivity of equality. Symmetry follows by setting  $c$  identical to  $a$ .

Brouwer introduced a whole range of notions relating to the notion of apartness, often with very idiosyncratic notations or terminologies. For example, in his (1927, 1950) a strict linear order  $a < b$  was introduced, with the symmetrization  $a < b \vee b < a$  corresponding to apartness, and the double negation  $\neg\neg a < b$  giving "the non-contradictory of the measurable natural order on the continuum."

A formal treatment of apartness relations began with Heyting's (1927) formalization of elementary intuitionistic geometry. In it, point and line equalities were replaced by corresponding apartnesses and the basic axiomatic properties studied. Heyting's little book (1956) presented intuitionistic axiomatizations of apartness and order relations. In the 1960s, such relations were used in different approaches to constructive analysis, as in Scott (1968) who established the notation we use. At the same time, there began a period when intuitionistic axiomatic systems were studied by the means of Kripke semantics, as in Smorynski (1973). Among the work of this period, there was an isolated paper by Van Dalen and Statman (1978) who studied the theory of apartness by syntactic, proof-theoretical means. Described as an "unpretentious little paper" in Smorynski's review (1980), it formulated the theory of apartness as an

extension of natural deduction in which the axioms for apartness were turned into nonlogical rules of inference. Van Dalen and Statman defined the theory of apartness as an extension of the theory of equality, with two primitive basic notions related by an axiom stating that the negation of apartness implies equality, and closure of apartness under substitution of equals. Next, they defined a hierarchy of extensions of the theory of equality, the first consisting in the addition of a stability axiom for equality, the  $n$ -th given by the addition of a generalized stability axiom  $S_n$  (see Section 4 for the details). By analyzing the structure of formal derivations in the system, Van Dalen and Statman were able to prove that if a formula containing no apartness symbol is derivable in the theory of apartness, then it is derivable in an appropriate  $n$ -th extension **SEQ** <sup>$n$</sup>  of the theory of equality. Next to this main result, they also showed by the use of Kripke models that each extension is proper, that is, all the stability axioms are needed to characterize the equality fragment of apartness.

In this paper we present, as an application of proof analysis in sequent calculi, a simplification of the argument of Van Dalen and Statman that was based on normalization in natural deduction, and replace their semantical arguments by proof-theoretical ones. As a background, we mention the work Negri (1999) that set out to solve a problem stemming directly from Van Dalen and Statman: To show that if equality is defined as the negation of apartness and if all the atomic formulas of a formula  $A$  appear negated, the derivability of  $A$  in the theory of apartness reduces to its derivability in the theory of equality. If we call formulas of the said kind “negatomic,” the result is that the quantifier-free theory of apartness is conservative over the theory of equality for negatomic formulas. This is in no way obvious, because the “split” axiom has a disjunction in a positive part.

In Negri (1999) and Negri and von Plato (1998), a systematic method was developed for extending sequent calculi with rules that correspond to mathematical axioms, in such a way that the structural properties of the logical sequent calculus are maintained in the extension. In particular, the admissibility of weakening, contraction, and cut permit to extend the methods of structural proof analysis of first-order logic to many mathematical theories.

By these results, the theory of apartness can be presented as a contraction- and cut-free sequent calculus. Equality is here a defined notion, given by the negation of apartness. A sequent calculus for the theory of defined equality is obtained by permitting rules with a specific succedent, in this case  $\perp$ , needed for translating into a rule the axiom of

transitivity. By using minimal logic, rather than intuitionistic logic, as a basic logical calculus, cut elimination is maintained. The reduction to minimal logic is justified by a conservativity result for sequents in the language of equality, that is, negatomic sequents (Theorem 21). Next we show how the conservativity result of Van Dalen and Statman follows in our approach by a straightforward analysis of cut-free derivations in the calculus for apartness. Finally, we obtain in a purely proof-theoretic way the result on proper extensions by showing the underderivability of the generalized stability axiom  $S_{n+1}$  in the  $n$ -th extension of the theory of equality.

## 2. Background on Proof Analysis

In Negri (1999) the intuitionistic theories of apartness and order were presented as contraction- and cut-free sequent calculi. The method, based on the conversion of axioms into inference rules, was then generalized in Negri and von Plato (1998, 2001) and in Negri (2003). We recall here the general ideas of the method and the main results. In addition, we show how the use of minimal logic permits a useful extension of the rule scheme for dealing with certain axioms that contain negations also in their negative part.

For extensions of intuitionistic predicate logic the starting point is the contraction- and cut-free multisuccedent sequent calculus **G3im** (see Negri and von Plato 2001, p. 108, for a table of rules). We shall here deal only with multisuccedent systems, and therefore leave out the suffix **m** from the name of this and derived systems. In **G3i** all the structural rules, weakening, contraction, and cut, are *admissible*, meaning that whenever their premisses are derivable, then so is their conclusion. Weakening and contraction are in addition *height-preserving* admissible, that is, whenever their premisses are derivable with derivation height bounded by  $n$ , then also their conclusion is derivable, with the same bound on the derivation height (the *height* of a derivation is its height as a tree, that is, the length of its longest branch). Moreover, the calculus enjoys height-preserving admissibility of substitution (see Chapters 3 and 4 of Negri and von Plato 2001 for detailed proofs).

The structural properties of **G3i** are maintained in extensions of the logical calculus with suitably formulated rules that represent axioms for specific theories. Formulas of the form  $P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$ , where the consequent is  $\perp$  if  $n = 0$  and all  $P_i, Q_j$  are atomic, called *regular* formulas, are converted into a schematic rule, called the *regular*

rule scheme, of the form

$$\frac{Q_1, \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}_{Reg}$$

where  $\overline{P}$  abbreviates the multiset  $P_1, \dots, P_m$ .

In order to maintain admissibility of contraction in the extensions with regular rules, the formulas  $P_1, \dots, P_m$  in the antecedent of the conclusion of the scheme have, as indicated, to be repeated in the antecedent of each of the premisses. In addition, whenever an instantiation of free parameters in atoms produces a duplication (two identical atoms) in the conclusion of a rule instance, say  $P_1, \dots, P, P, \dots, P_m, \Gamma \Rightarrow \Delta$ , there is a corresponding duplication in each premiss and in the conclusion of the rule. The *closure condition* imposes the requirement that the rule with the duplication  $P, P$  contracted into a single  $P$  be added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all, so the condition is unproblematic.

The main result for such extensions is the following (Theorems 6.2.1, 6.2.2., 6.2.3 of Negri and von Plato 2001):

**Theorem 1** *The structural rules of weakening, contraction, and cut are admissible in all extensions of **G3i** with the regular rule scheme *Reg* and satisfying the closure condition. Weakening and contraction are moreover height-preserving admissible.*

In classical logic, each universal axiom can be converted, through reduction to conjunctive normal form, to a conjunction of regular formulas, and therefore all universal axioms can be converted into regular rules. By using intuitionistic logic, we have instead a limitation to those axioms that are constructively equivalent to a conjunction of regular formulas. In particular, implications the antecedents of which are implications, or more specifically, negations, are left out by the general treatment. There is however a special case of implication of this form that is met when treating equality as the negation of apartness: If equality  $a = b$  is defined as  $\neg a \neq b$ , then transitivity (in the Euclidean form) is the axiom

$$\neg a \neq c \ \& \ \neg b \neq c \supset \neg a \neq b$$

that does not follow the form of a regular formula, but is of the form

$$\neg P_1 \& \dots \& \neg P_n \supset \neg Q$$

It can be converted into the rule

$$\frac{P_1, \Gamma \Rightarrow \perp \quad \dots \quad P_n, \Gamma \Rightarrow \perp}{Q, \Gamma \Rightarrow \perp}$$

or, more precisely, taking into account the requirements for the admissibility of weakening and contraction, into

$$\frac{P_1, Q, \Gamma \Rightarrow \perp \quad \dots \quad P_n, Q, \Gamma \Rightarrow \perp}{Q, \Gamma \Rightarrow \Delta, \perp} \text{Reg}_{\perp}$$

Observe that the succedent of the above scheme is not an arbitrary multiset as required in the regular rule scheme, but the logical constant  $\perp$ . In general, cut elimination would fail in the presence of a scheme of this form; However, it is maintained in extensions of *minimal*, rather than intuitionistic, logic.

In general, a calculus for minimal logic is obtained from a calculus for intuitionistic logic by removing the axiom of *ex falso quodlibet*. In **G3** calculi, the identity axiom is restricted to atomic formulas, thus, in order to obtain a **G3** calculus for minimal logic, rule  $L \perp$  is removed from the rules of **G3i**, but its special instance  $\perp, \Gamma \Rightarrow \Delta, \perp$  is added. The calculus thus obtained is called **G3m**. We have:

**Theorem 2** *The structural rules of weakening, contraction, and cut are admissible in all extensions of **G3m** with rules following the regular rule scheme  $\text{Reg}$  or the scheme  $\text{Reg}_{\perp}$  and satisfying the closure condition. Weakening and contraction are moreover height-preserving admissible.*

**Proof.** Admissibility of weakening and contraction presents no additional difficulties as compared to the proof for extensions of **G3i** with the regular rule scheme. As for admissibility of cut, we have to analyze those new cases in which the active formula of the scheme  $\text{Reg}_{\perp}$  is the cut formula

$$\frac{\frac{P_1, Q, \Gamma \Rightarrow \perp \quad \dots \quad P_n, Q, \Gamma \Rightarrow \perp}{Q, \Gamma \Rightarrow \Delta, \perp} \text{Reg}_{\perp} \quad \perp, \Gamma' \Rightarrow \Delta'}{Q, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{Cut}$$

If  $\perp$  is not principal in the right premiss of cut, the cut is permuted with the rule concluding the right premiss of cut, and the height decreased. If it is principal, the right premiss is necessarily an initial sequent of the form  $\perp, \Gamma' \Rightarrow \Delta'', \perp$  and the conclusion of cut is obtained by weakening from the left premiss.  $\square$

The analysis of cut-free derivations in regular extensions of **G3i** or in extensions of **G3m** with rules following the rule scheme  $Reg_{\perp}$  is based on the **weak subformula property**: *All formulas in a derivation are subformulas of the endsequent or atomic formulas.* This property ensures many of the consequences of the full subformula property and can often be strengthened by the **subterm property**: *All terms in a derivation are terms in the conclusion.* Thus, only a bounded number of distinct atomic formulas appear.

### 3. Sequent Calculi for the Theories of Apartness and Order

The apartness relation is an intuitionistic basic notion that represents “a positive analogue of the inequality relation on the continuum” (Van Dalen and Statman 1978, p. 95). Inequality thus becomes a basic notion in intuitionistic axiomatics. As explained in the introduction, we shall use the symbol  $\neq$  for apartness, and denote ‘ $a$  is apart from  $b$ ’ by  $a \neq b$ . The negation of equality,  $a = b \supset \perp$ , will be denoted by  $\neg a = b$ . Van Dalen and Statman, instead, use the symbol  $\#$  for apartness and  $\neq$  as an abbreviation for the negation of equality.

The axioms for apartness are<sup>1</sup>

$$Irref \quad \neg a \neq a \quad Split \quad a \neq b \supset a \neq c \vee b \neq c$$

Following the method of extension with rules recalled in Section 2, we have:

**Definition 3** *The sequent calculus for the theory of apartness **G3AP** is obtained by adding to **G3i** the following rules*

$$\frac{}{a \neq a, \Gamma \Rightarrow \Delta} Irref \quad \frac{a \neq c, a \neq b, \Gamma \Rightarrow \Delta \quad b \neq c, a \neq b, \Gamma \Rightarrow \Delta}{a \neq b, \Gamma \Rightarrow \Delta} Split$$

As an application of the general result recalled in Theorem 1 we have:

**Theorem 4** *The structural rules of weakening, contraction, and cut are admissible in **G3AP**.*

The axioms for the theory of defined equality are

$$Irref \quad \neg a \neq a \quad Trans \quad \neg a \neq c \ \& \ \neg b \neq c \supset \neg a \neq b$$

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<sup>1</sup>In the literature, the split axiom in the form  $a \neq b \supset a \neq c \vee c \neq b$  is called also *co-transitivity*.



The latter is not a regular formula, because there are implications (negations) in the antecedent of an implication. However, as we have seen in Section 2, such an axiom can be added to sequent calculus in the form of a rule and admissibility of the structural rules maintained, if minimal logic is used as a logical basis.

**Definition 5** *The calculus **G3EQ** is obtained by adding to the multi-succedent calculus for minimal logic **G3m** the rules*

$$\frac{}{a \neq a, \Gamma \Rightarrow \Delta} Irref \quad \frac{a \neq c, a \neq b, \Gamma \Rightarrow \perp \quad b \neq c, a \neq b, \Gamma \Rightarrow \perp}{a \neq b, \Gamma \Rightarrow \Delta, \perp} Split_{\perp}$$

By Theorem 2 we have:

**Theorem 6** *The structural rules are admissible in **G3EQ**. Weakening and contraction are height-preserving admissible.*

As a corollary of cut elimination we have:

**Corollary 7** *The calculi **G3AP** and **G3EQ** satisfy the weak subformula property: Each formula in a derivation is a subformula of the end-sequent or an atomic formula.*

The axioms of reflexivity, transitivity, and stability,  $\neg\neg a = b \supset a = b$ , are derivable in the calculus **G3EQ**, thus the calculus is complete for the theory of stable equality.

## 4. The Equality Fragment of Apartness

In Negri (1999) it was proved that, for the propositional part, the theory of apartness is conservative over the theory of equality defined as the negation of apartness. In other words, the axiom of stability added to the theory of equality suffices for ensuring conservativity in the propositional fragment.

In order to isolate “the equality fragment of **AP**”, Van Dalen and Statman identify in  $\forall z(\neg z = a \vee \neg z = b)$  an important kind of inequality relation, stronger than the ordinary inequality. They give the following inductive definition:

$$a \neq_0 b \equiv \neg a = b \quad a \neq_{n+1} b \equiv \forall z(z \neq_n a \vee z \neq_n b)$$

Parallel to these generalized inequality relations, a sequence of generalized stability axioms, stronger than ordinary stability, is given:

$$S_n \equiv \neg a \neq_n b \supset a = b$$

Defining as **SEQ** $^\omega$  the  $\omega$ -stable theory of equality **EQ** +  $\{S_n | n \in \omega\}$ , Van Dalen and Statman (1978, p. 105) prove, by means of natural deduction, the following result:

Suppose that  $B$  is an equality formula,  $X$  is a set of equality formulæ and  $Y$  is a set of apartnesses.

Set  $Y(n) = \{t_1 \neq_n t_2 : t_1 \# t_2 \in Y\} [\dots]$

If  $X \cup Y \vdash_{AP} B$ , then  $X \cup Y(n) \vdash_{SEQ^\omega} B$  for some  $n$ .

Therefore, the theory **AP** is conservative over **SEQ** $^\omega$ .

In order to achieve a fully proof-theoretical account of the results by Van Dalen and Statman, we give a sequent calculus in which the inequality relations  $\neq_n$  and the generalized stability axioms are added in the form of nonlogical rules to the theory of defined equality:

**Definition 8** The calculus **G3EQ** $^n$  for  $n > 0$  is obtained by adding to **G3EQ** the rules  $Split_m$  and  $Def_m$  (for all  $m$ ) and the rule  $S_n$

$$\frac{a \neq_{m-1} c, a \neq_m b, \Gamma \Rightarrow \Delta \quad b \neq_{m-1} c, a \neq_m b, \Gamma \Rightarrow \Delta}{a \neq_m b, \Gamma \Rightarrow \Delta} Split_m$$

$$\frac{\Gamma \Rightarrow a \neq_{m-1} x, b \neq_{m-1} x}{\Gamma \Rightarrow \Delta, a \neq_m b} Def_m \quad \frac{a \neq_n b, a \neq b, \Gamma \Rightarrow \perp}{a \neq b, \Gamma \Rightarrow \Delta, \perp} S_n$$

where, in rule  $Def_m$ , the variable  $x$  must not appear free in the conclusion.

Note that, in the theory of defined equality,  $a \neq_0 b \equiv \neg \neg a \neq b$ , and therefore **G3EQ** $^0 \equiv \mathbf{G3EQ} + S_0 \equiv \mathbf{G3EQ}$ .

**Lemma 9** The sequents

$$a \neq_{m+1} b \Rightarrow \forall x (a \neq_m x \vee b \neq_m x) \text{ and } \forall x (a \neq_m x \vee b \neq_m x) \Rightarrow a \neq_{m+1} b$$

are derivable in **G3EQ** $^n$ .

**Proof.** By means of  $Split_{m+1}$  and  $Def_{m+1}$  respectively.  $\square$

**Lemma 10** The sequent  $\neg a \neq_n b \Rightarrow \neg a \neq b$  is derivable in **G3EQ** $^n$ .

**Proof.** By induction on  $n$ . If  $n = 0$ , by the definition of  $\neq_0$  we have the sequent  $\neg \neg \neg a \neq b \Rightarrow \neg a \neq b$ . If  $n > 0$ ,  $S_n$  is used.  $\square$

**Lemma 11** *All sequents of the form  $a \neq_n b, \Gamma \Rightarrow \Delta, a \neq_n b$  are derivable in  $\mathbf{G3EQ}^n$ .*

**Proof.** By induction on  $n$ . For  $n = 0$ , use the definition of  $\neq_0$  and steps of propositional logic. For  $n > 0$ , apply the inductive hypothesis and use  $\text{Split}_n$  and  $\text{Def}_n$ .  $\square$

As for  $\mathbf{G3EQ}$ , it is routine to prove:

**Theorem 12** *The rules of weakening and contraction are admissible and height-preserving in  $\mathbf{G3EQ}^n$ .*

Formulas of the form  $a \neq_n b$  are by definition compound formulas, but here they are regarded as atomic and will be called **inequality formulas**.

The proof of cut elimination follows the usual structure of the proof for extensions of  $\mathbf{G3}$  calculi with nonlogical rules, with one extra case due to the presence of a detour cut on inequality formulas. The proof is by induction on the weight of the cut formula with subinduction on cut height. The weight is defined as usual for compound formula, with an extra degree of complexity for formulas of the form  $a \neq_n b$ . The definition of weight of  $a \neq_n b$  takes into account the inductive definition of the relations  $\neq_n$  and is made consistent with the definition of weight of formulas (cf. Negri and von Plato 2001, pp. 30 and 67).

**Definition 13** *The weight of an inequality formula is*

$$w(a \neq_0 b) \equiv 3 \quad w(a \neq_{n+1} b) \equiv 2w(a \neq_n b) + 2$$

For our purposes, it is enough to note that the definition of weight for an inequality formula is an increasing function of  $n$ .

Rules  $\text{Split}_n$  and  $\text{Def}_n$  emulate the left and right universal quantifier rules. A substitution lemma is therefore needed for dealing with detour cuts on inequality formulas. The proof is similar to the proof of Lemma 4.1.2. in Negri and von Plato (2001).

**Lemma 14** *If  $\Gamma \Rightarrow \Delta$  is derivable in  $\mathbf{G3EQ}^n$  and  $t$  is free for  $y$  in  $\Gamma, \Delta$ , then  $\Gamma(t/y) \Rightarrow \Delta(t/y)$  is derivable in  $\mathbf{G3EQ}^n$  with the same derivation height.*

**Theorem 15** *The rule of cut is admissible in  $\mathbf{G3EQ}^n$ .*

**Proof.** By induction on the weight of the cut formula with subinduction on cut height. Let us consider only the cases introduced by the new nonlogical rules. The cases in which neither premiss of cut is concluded by  $Def_n$  are handled as in the proof of Theorem 2.

If the left premiss of cut is a conclusion of  $Def_n$  with the cut formula not principal in it, the conclusion of cut is obtained by the rule and several weakenings. If the left premiss is a conclusion of  $Def_n$  with the cut formula principal in it but not in the right premiss, we permute the cut upwards in the derivation; otherwise the right premiss is a conclusion of  $Split_n$  and the derivation is

$$\frac{\frac{\Gamma \Rightarrow a \neq_{n-1} x, b \neq_{n-1} x}{\Gamma \Rightarrow \Delta, a \neq_n b} Def_n \quad \frac{a \neq_{n-1} c, a \neq_n b, \Gamma' \Rightarrow \Delta' \quad b \neq_{n-1} c, a \neq_n b, \Gamma' \Rightarrow \Delta'}{a \neq_n b, \Gamma' \Rightarrow \Delta'} Split_n}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Cut$$

First we cut the conclusion of  $Def_n$  with the left premiss of  $Split_n$

$$\frac{\frac{\Gamma \Rightarrow a \neq_{n-1} x, b \neq_{n-1} x}{\Gamma \Rightarrow \Delta, a \neq_n b} Def_n \quad a \neq_{n-1} c, a \neq_n b, \Gamma \Rightarrow \Delta}{a \neq_{n-1} c, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Cut \quad 1.$$

Then we cut the conclusion of  $Def_n$  with the right premiss of  $Split_n$

$$\frac{\frac{\Gamma \Rightarrow a \neq_{n-1} x, b \neq_{n-1} x}{\Gamma \Rightarrow \Delta, a \neq_n b} Def_n \quad b \neq_{n-1} c, a \neq_n b, \Gamma \Rightarrow \Delta}{b \neq_{n-1} c, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Cut \quad 2.$$

Observe that both cuts have reduced height. Finally, we use the conclusions of the preceding derivations as follows:

$$\frac{\frac{\Gamma \Rightarrow a \neq_{n-1} c, b \neq_{n-1} c \quad a \neq_{n-1} c, \Gamma, \Gamma' \stackrel{1}{\Rightarrow} \Delta, \Delta'}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b \neq_{n-1} c} Cut \quad \frac{b \neq_{n-1} c, \Gamma, \Gamma' \stackrel{2}{\Rightarrow} \Delta, \Delta'}{\Gamma, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'} Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Ctr^*$$

where  $Ctr^*$  stands for several contractions and the sequent

$$\Gamma \Rightarrow a \neq_{n-1} c, b \neq_{n-1} c$$

is obtained by Lemma 14 from the premiss of  $Def_n$ . Note that the two cuts are reduced in weight.

Finally, if the right premiss is a conclusion of  $Def_n$ , the cut formula is not principal in it and we apply the inductive hypothesis to the premiss of this rule.  $\square$

**Corollary 16** *The calculus G3EQ enjoys the weak subformula property: Each formula in a derivation is a subformula of the endsequent, or an atomic, or an inequality formula.*



that can be replaced by

$$\frac{\frac{\frac{\frac{\frac{\frac{\perp, \neg P, \Gamma' \Rightarrow \perp}{\perp, \Gamma' \Rightarrow \neg\neg P} R \supset}{\vdots}}{\neg\neg P, \Gamma \Rightarrow \neg\neg P}}{\neg P \Rightarrow \neg\neg\neg P} \quad \frac{\frac{\perp, \Gamma \Rightarrow \Delta}{\neg\neg P, \Gamma \Rightarrow \Delta} L \supset}{\neg P, \Gamma \Rightarrow \Delta} Cut$$

where the sequent  $\neg P \Rightarrow \neg\neg\neg P$  is derivable in minimal logic.  $\square$

In  $\mathbf{G3EQ}^n$ , we can prove, by induction on  $n$  and application of rule  $Def_n$ :

**Lemma 22** *The sequent  $\perp, \Gamma \Rightarrow \Delta, a \neq_n b$  is derivable in the calculus  $\mathbf{G3EQ}^n$  for all  $n$ .*

**Corollary 23** *The zero-premiss rule  $Irref_n$*

$$\frac{}{a \neq_n a, \Gamma \Rightarrow \Delta} Irref_n$$

*is admissible in  $\mathbf{G3EQ}^n$  for  $\Delta$  containing an inequality or negatomic formula.*

**Proof.** By induction on  $n$ . If  $n = 0$ , use the definition of  $\neq_0$ ,  $Irref$ , Lemmas 20 and 22, and steps of propositional logic. If  $n > 0$ , apply the inductive hypothesis and  $Split_n$ .  $\square$

**Corollary 24** *The sequent  $\neg a \neq b \Rightarrow \neg a \neq_n b$  is derivable in  $\mathbf{G3EQ}^n$ .*

**Proof.** By induction on  $n$ . For  $n = 0$ , by steps of propositional logic we derive the sequent  $\neg\neg a \neq b, \neg a \neq b \Rightarrow \perp$ , which is equivalent to  $a \neq_0 b, \neg a \neq b \Rightarrow \perp$  by definition of  $\neq_0$ . We obtain then  $\neg a \neq b \Rightarrow \neg a \neq_0 b$  by an application of  $R \supset$ . Next, let us suppose by inductive hypothesis that the sequent  $\neg a \neq b \Rightarrow \neg a \neq_n b$  is derivable; then by invertibility of  $R \supset$  also  $a \neq_n b, \neg a \neq b \Rightarrow \perp$  is derivable. We have the following derivation:

$$\frac{\frac{a \neq_n b, a \neq_{n+1} b, \neg a \neq b \Rightarrow \perp \quad \frac{\frac{}{b \neq_n b, a \neq_{n+1} b, \neg a \neq b \Rightarrow \perp} Irref_n}{a \neq_{n+1} b, \neg a \neq b \Rightarrow \perp} Split_{n+1}}{\neg a \neq b \Rightarrow \neg a \neq_{n+1} b} R \supset$$

where the sequent  $a \neq_n b, a \neq_{n+1} b, \neg a \neq b \Rightarrow \perp$  is obtained by height-preserving weakening.  $\square$

We are now in a position to prove the conservativity theorem that characterizes the equality fragment of apartness. The proof is similar to the proof presented in Troelstra and Schwichtenberg (2000) that in turn was inspired by the method in Negri (1999). The central idea is the replacement of apartness atoms, for each step of *Split* in the derivation, with appropriate inequality formulas. The definition of positive and negative occurrences of a symbol in a formula is standard.

**Definition 25** *Let  $\Gamma \Rightarrow \Delta$  be a sequent in the language of apartness. Then  $\Gamma(n) \Rightarrow \Delta(n)$  denotes the sequent obtained by replacing all positive occurrences of  $\neq$  in  $\Gamma$  and all negative occurrences of  $\neq$  in  $\Delta$  by  $\neq_n$  and by replacing all negative occurrences of  $\neq$  in  $\Gamma$  and all positive occurrences of  $\neq$  in  $\Delta$  by  $\neq_0$ .*

**Lemma 26** *If the sequent  $\Gamma(n) \Rightarrow \Delta(n)$  is derivable in  $\mathbf{G3EQ}^n$ , then also  $\Gamma(n+1) \Rightarrow \Delta(n+1)$  is.*

**Proof.** For each replaced atom  $a \neq_n b$ , use the derivable sequent

$$a \neq_{n+1} b \Rightarrow a \neq_n b \quad \square$$

**Theorem 27** *If  $\Gamma \Rightarrow \Delta$  is derivable in  $\mathbf{G3mAP}$ , then there exists  $n$  such that  $\Gamma(n) \Rightarrow \Delta(n)$  is derivable in  $\mathbf{G3EQ}^n$ .*

**Proof.** By induction on the height  $h$  of the given derivation. If  $h = 0$  and  $\Gamma \Rightarrow \Delta$  is of the form  $a \neq b, \Gamma' \Rightarrow \Delta', a \neq b$ , observe that, by Lemma 11,  $a \neq_n b, \Gamma'(n) \Rightarrow \Delta'(n), a \neq_n b$  is derivable in  $\mathbf{G3EQ}^n$  for all  $n$  and a cut with  $a \neq_n b \Rightarrow a \neq_0 b$  gives a derivation of the sequent  $\Gamma(n) \Rightarrow \Delta(n)$ . If  $\Gamma \Rightarrow \Delta$  is of the form  $\perp, \Gamma' \Rightarrow \Delta', \perp$ , then  $\perp, \Gamma'(n) \Rightarrow \Delta'(n), \perp$  is for all  $n$  an initial sequent in  $\mathbf{G3EQ}^n$  as well. If it is an instance of *Irref*,  $a \neq a, \Gamma' \Rightarrow \Delta$ , the sequent  $a \neq_n a, \Gamma'(n) \Rightarrow \Delta(n)$  is by Corollary 23 an admissible instance of *Irref* <sub>$n$</sub>  in  $\mathbf{G3EQ}^n$ .

If the last rule in the derivation is a logical rule, apply the inductive hypothesis to the premiss(es) of the rule and then the rule.

If the last rule is *Split*, with premisses

$$a \neq c, a \neq b, \Gamma' \Rightarrow \Delta$$

and

$$b \neq c, a \neq b, \Gamma' \Rightarrow \Delta,$$

by the inductive hypothesis there exist  $n$  and  $m$  such that the sequents  $a \neq_n c, a \neq_n b, \Gamma'(n) \Rightarrow \Delta(n)$  and  $a \neq_m c, a \neq_m b, \Gamma'(m) \Rightarrow \Delta(m)$  are derivable in  $\mathbf{G3EQ}^n$  and in  $\mathbf{G3EQ}^m$ , respectively. Choose  $p \geq n, m$ . Then, by Lemma 26,  $a \neq_p c, a \neq_p b, \Gamma'(p) \Rightarrow \Delta(p)$  and  $a \neq_p c, a \neq_p b, \Gamma'(p) \Rightarrow \Delta(p)$  are derivable in  $\mathbf{G3EQ}^p$ . By using the derivable  $a \neq_{p+1} b \Rightarrow a \neq_p b$ , the sequents

$$a \neq_p c, a \neq_{p+1} b, \Gamma'(p) \Rightarrow \Delta(p)$$

and

$$a \neq_p c, a \neq_{p+1} b, \Gamma'(p) \Rightarrow \Delta(p)$$

are derivable in  $\mathbf{G3EQ}^p$ . By *Split* <sub>$p+1$</sub>  and Lemma 26 we derive

$$a \neq_{p+1} b, \Gamma'(p+1) \Rightarrow \Delta(p+1)$$

□

We observe that it is necessary to prove the above theorem for arbitrary sequents, and not just negatomic ones, because the premisses of a logical rule with negatomic conclusion need not be negatomic.

**Corollary 28** *If a negatomic sequent  $\Gamma \Rightarrow \Delta$  is derivable in  $\mathbf{G3AP}$ , then it is derivable in  $\mathbf{G3EQ}^\omega$ .*

**Proof.** By Lemma 20 a negatomic sequent derivable in  $\mathbf{G3AP}$  is derivable in  $\mathbf{G3mAP}$ . By Theorem 27,  $\Gamma(n) \Rightarrow \Delta(n)$  is derivable in  $\mathbf{G3EQ}^n$  for some  $n$ , and using the equivalence  $\neg a \neq_n b \Leftrightarrow \neg a \neq b$  (Lemma 10 and Corollary 24),  $\Gamma(n) \Rightarrow \Delta(n)$  is equivalent to the (negatomic) sequent  $\Gamma \Rightarrow \Delta$ . □

## 5. Underivability through Proof Analysis in $\mathbf{G3EQ}^n$

We shall prove that the whole system  $\mathbf{G3EQ}^\omega$  is needed in order to deal with the equality fragment of the theory of apartness, because for no  $n$  does  $\mathbf{G3EQ}^n$  suffice: Each theory  $\mathbf{G3EQ}^{n+1}$  is, in fact, a proper extension of the theory  $\mathbf{G3EQ}^n$ .

We establish first the subterm property for  $\mathbf{G3EQ}^n$ . We recall that a derivation is *minimal* if no local shortenings through deletion of steps are possible: For example, if a conclusion of a rule becomes an instance of an initial sequent, the derivation can be shortened by pruning the branch above that sequent; If a rule has the effect of a contraction, the rule can be deleted by height-preserving contraction.



**Theorem 29** *In a minimal derivation in  $\mathbf{G3EQ}^n$ , all terms are eigen-variables or terms in the conclusion.*

**Proof.** We show that in a minimal derivation all middle terms of  $Split_\perp$  and of  $Split_n$  are terms in the conclusion.

Consider the last occurrence of a term  $c$  removed by  $Split_\perp$

$$\frac{a \neq c, a \neq b, \Gamma \Rightarrow \perp \quad b \neq c, a \neq b, \Gamma \Rightarrow \perp}{a \neq b, \Gamma \Rightarrow \Delta, \perp} Split_\perp$$

Trace, along either branch, the atom containing  $c$  and activated by the rule in which  $a \neq c$  (or  $b \neq c$ ) is principal. (Observe that if  $a \neq c$  or  $b \neq c$  is never principal, the atom can be removed together with the step of  $Split_\perp$  and thus the derivation shortened). If this rule is another instance of  $Split_\perp$ , the atom has the form  $c \neq d$ , if it is  $S_n$  it is of the form  $a \neq_n c$ . This latter atom can be principal in  $Split_n$ , and we continue by tracing the active atom of the rule, the one that contains  $c$ . Eventually we find an atom  $f \neq c$  in an initial sequent or in *Irref*. (Observe two things here: 1. By Lemma 11 we need not consider initial sequents for inequality formulas. 2. If another atom, not containing  $c$ , is principal, then also the conclusion of the rule removing  $f \neq c$  is an instance of *Irref* or of an initial sequents, so the derivation can be shortened).

If we have the atom  $f \neq c$  in an initial sequent, the succedent contains  $c$ , hence also the conclusion does. If  $f \neq c$  is principal in *Irref*, we have  $f \equiv c$ . Consider the step of  $Split_\perp$  that removes the atom  $c \neq c$  (observe that this step must be above the instance of  $Split_\perp$  from which we started our analysis, since by hypothesis we were considering the last occurrence of the term  $c$ ). Then also the other premiss must contain an atom with the term  $c$ . Trace then along the other branch the atoms containing  $c$  activated in the derivation. Eventually, we find again the term  $c$  in an atom that is principal in an initial sequent or in *Irref*. In the former case, we find  $c$  in the conclusion of the derivation, in the latter we repeat the procedure by tracing along the other premiss of the step of  $Split_\perp$ . Since the derivation is finite, this cannot go on indefinitely, that is, at some point we find the term  $c$  in an atom that is principal in an initial sequent.

If a term is removed by  $Split_n$ , consider its last occurrence in a step of the form

$$\frac{a \neq_{n-1} c, a \neq_n b, \Gamma \Rightarrow \Delta \quad b \neq_{n-1} c, a \neq_n b, \Gamma \Rightarrow \Delta}{a \neq_n b, \Gamma \Rightarrow \Delta} Split_n$$

Now proceed as for  $Split_{\perp}$ , by tracing the activated atoms containing  $c$ . Eventually, an apartness atom containing  $c$  is reached and the proof proceeds as above.  $\square$

**Theorem 30** *The sequent  $\neg a \neq_{n+1} b \Rightarrow \neg a \neq b$  is not derivable in the calculus **G3EQ**<sup>n</sup>.*

**Proof.** We search for a minimal derivation of the sequent. By the invertibility of  $R \supset$ , the sequent  $\neg a \neq_{n+1} b \Rightarrow \neg a \neq b$  is derivable if and only if  $\neg a \neq_{n+1} b, a \neq b \Rightarrow \perp$  is derivable. The last rule in the derivation of this sequent can be  $Split_{\perp}$ , where, by Theorem 29, the middle term cannot be a new term. If the middle term is  $b$ , the derivation is

$$\frac{\neg a \neq_{n+1} b, a \neq b, a \neq b \Rightarrow \perp \quad \frac{}{\neg a \neq_{n+1} b, b \neq b, a \neq b \Rightarrow \perp} Irref}{\neg a \neq_{n+1} b, a \neq b \Rightarrow \perp} Split_{\perp}$$

A duplication of the formula  $a \neq b$  is found in a premiss and application of height-preserving contraction gives the conclusion in one step less, against the hypothesis of minimality; analogously for middle term  $a$ .

The last rule can be also  $S_n$

$$\frac{\neg a \neq_{n+1} b, a \neq_n b, a \neq b \Rightarrow \perp}{\neg a \neq_{n+1} b, a \neq b \Rightarrow \perp} S_n$$

with  $\neg a \neq_{n+1} b, a \neq_n b, a \neq b \Rightarrow \perp$  conclusion of  $L \supset$

$$\frac{\neg a \neq_{n+1} b, a \neq_n b, a \neq b \Rightarrow a \neq_{n+1} b \quad \perp, a \neq_n b, a \neq b \Rightarrow \perp}{\neg a \neq_{n+1} b, a \neq_n b, a \neq b \Rightarrow \perp} L \supset$$

Continuing with  $L \supset$  would lead to a loop, because it reproduces the conclusion in its left premiss. Application of rule  $Def_{n+1}$  gives

$$\neg a \neq_{n+1} b, a \neq_n b, a \neq b \Rightarrow a \neq_n x, b \neq_n x.$$

By iterating  $L \supset$ ,  $Split_n$ ,  $Def_n$ , we obtain sequents where inequality formulas with the same index  $n$  appear on the left and right hand-side, but these are not derivable since their terms do never coincide because the variable in  $Def_n$  is fresh.

Finally, if the sequent  $\neg a \neq_{n+1} b, a \neq_n b, a \neq b \Rightarrow \perp$  is a conclusion of  $Split_n$ , by Theorem 29 the middle term can be only  $a$  or  $b$ . If the middle term is  $b$ , the right premiss  $\neg a \neq_{n+1} b, b \neq_{n-1} b, a \neq_n b, a \neq b \Rightarrow \perp$  is conclusion of  $Irref_{n-1}$ , thus derivable by Corollary 23. The left

premiss is the sequent  $\neg a \neq_{n+1} b, a \neq_{n-1} b, a \neq_n b, a \neq b \Rightarrow \perp$ . By iterating this process we find the sequent  $\neg a \neq_{n+1} b, a \neq_0 b, \dots, a \neq_n b, a \neq b \Rightarrow \perp$ ; analogously if the middle term is  $a$ .

If we continue with  $L \supset$  with left premiss the sequent  $\neg a \neq_{n+1} b, a \neq_0 b, \dots, a \neq_n b, a \neq b \Rightarrow a \neq_{n+1} b$ , proof search fails for the reasons seen before. By definition of  $a \neq_0 b$ , the last rule can be also  $L \supset$  with

$$\neg a \neq_{n+1} b, a \neq_0 b, \dots, a \neq_n b, a \neq b \Rightarrow \neg a \neq b$$

as left premiss, but now proof search continues as from the beginning with the addition of the formulas  $a \neq_0 b, \dots, a \neq_n b, a \neq b$  in the antecedent.

□

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